

MATHEMATICAL NUTS

FOR LOVERS OF MATHEMATICS

BY

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INSURANCE COMPANY, NASHVILLE, TENN., AND FORMERLY
PROFESSOR OF MATHEMATICS IN DAVID LIPS-
COMB COLLEGE, NASHVILLE, TENN.

1936 EDITION
(REVISED)

PRICE \$3.50, POSTPAID

SEND ALL ORDERS TO
SAMUEL I. JONES, PUBLISHER
1122 BELVEDERE DRIVE, NASHVILLE, TENN., U.S.A.

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PRINTED AND BOUND IN THE U. S. A. BY
KINGSPORT PRESS, INC., KINGSPORT, TENN.

“As the drill will not penetrate the granite unless kept to the work hour after hour, so the mind will not penetrate the secrets of mathematics unless held long and vigorously to the work. As the sun’s rays burn only when concentrated, so the mind achieves mastery in mathematics, and indeed in every branch of knowledge, only when its possessor hurls all his forces upon it. Mathematics, like all the other sciences, opens its doors to those only who knock long and hard. No more damaging evidence can be adduced to prove the weakness of character than for one to have aversion to mathematics; for whether one wishes so or not, it is nevertheless true; that to have aversion for mathematics means to have aversion to accurate, painstaking, and persistent hard study, and to have aversion to hard study is to fail to secure a liberal education, and thus fail to compete in that fierce and vigorous struggle for the highest and the truest and the best in life which only the strong can hope to secure.” — B. F. FINKEL.

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INTRODUCTION

There are many varieties of nuts and various ways of classifying them. Mr. Ralph Parlette in "It's Up to You" classified them as Little nuts and Big nuts. He says, "I hold in my hand this glass jar containing little white beans and big black walnuts. I mix them all up. Then I shake the jar. They un-mix. The walnuts go to the top and the little beans go to the bottom." "This little jar is a picture of what is going on everywhere in this world all the time. The world is just a big jar of life. All the people are in the jar getting jarred around all the time. All kinds of people are in the jar of life. The jar goes on shaking all the time. It never stops shaking. We are either Big nuts or Little nuts. Everybody is doing one of three things: Holding his place, rattling down, or shaking up."

The object of this work is to help its readers to occupy a higher position in the jar of life. We go up by our own growing. Others can lend a helping hand, but it is up to you! Are you shaking up or rattling down?

No better food for growth can be found than nuts. They are very nutritious. Instead, however, of classifying them into Big nuts and Little nuts we will classify them according to hardness of shell, or difficulty of cracking them, *i.e.*, soft, medium, and hard.

The author wishes to acknowledge his indebtedness to the editors of "School Science and Mathematics" and *The American Mathematical Monthly* for permission to use a number of problems and solutions which appeared in these most excellent journals. He wishes to thank his friend

Mr. J. F. Jones of Mt. Juliet, Tenn., for contributing some very interesting problems. Also he acknowledges his indebtedness to his many friends in various parts of the country whose names are too numerous to be mentioned here. Some gems were taken from "The School Visitor," which was published many years ago by Mr. John S. Royer. While many of the problems, recreations, solutions, etc., are original with the author, his aim has been to present the most useful and entertaining ones to be had.

This volume is intended as a companion book to "Mathematical Wrinkles," which has been highly commended by High School, College, and University professors in all parts of the world. It is the hope of the author that this volume will be very useful to both teachers and students and that this form of mental diversion will be welcomed by all lovers of mathematics as a mental tonic for eliminating the "sleepy brain" by sweeping away the cobwebs.

SAMUEL I. JONES.

NASHVILLE, TENN.

NUTS

MATHEMATICAL NUTS

NUTS FOR YOUNG AND OLD

1. A in five hours a sum can count,
Which B can in eleven ;
How much more then is the amount
They both can count in seven?
2. How many equilateral triangles can be formed from
5 nine matches, the side of each triangle being the length
of a match?
3. Given eight sticks, four of them being exactly half
the length of the others. Required to inclose three equal
squares with them.

4. Write an even number, using only odd digits.

A FISH QUESTION

5. Ten fish I caught without an eye,
And nine without a tail ;
Six had no head, and half of eight
I weighed upon the scale.
Now who can tell me, as I ask it,
How many fish were in my basket?
6. Find a number of two digits which is equal to twice
the product of those digits.
7. Write 55, using only five fours.
8. Write 1, using each of the nine digits only once.
9. Express 1, using five nines in three or more ways.

10. A train leaves New York for Chicago, traveling at the rate of 100 miles an hour. Another train leaves Chicago an hour later, traveling at 75 miles an hour. When the two trains meet, which one is nearest to New York?

11. Express 12 by four figures each the same.

12. If five times four are thirty-three,
What will the fourth of twenty be?

13. With the digits of the following number

7 7 7, 5 5 5, 3 3 3, 1 1 1

form six numbers whose sum is 20, using each digit once and only once.

14. Can you justify the following solution in simple addition?

$$\begin{array}{r}
 3414 \\
 340 \\
 74813 \\
 \hline
 433748.13
 \end{array}$$

Although the process seems to be incorrect and the result impossible both can be shown to be correct.

15. Express 100 by using four fives in five different ways.

16. Given 7 matches arranged as follows: $\frac{1}{VII}$, i.e. $\frac{1}{7}$. ^{XX}

The problem is to transform this fraction into one whose value shall be $\frac{1}{3}$ without adding or removing any matches.

17. A farmer surrounded his square melon patch with a ditch 10 feet wide and 20 feet deep. He filled this ditch with water and considered the patch well protected. However, with nothing but two $9\frac{1}{2}$ -foot planks, thieves bridged the ditch and stole the melons. How did they do it?

18. Express 100 by using six matches.

19. Express 10 by using 9 matches, not expressing it TEN.

20. Given 44 matches. Required to arrange them so as to express a fraction whose value is unity.

21. Write 1000 by using seven 9's.

22. Two monkeys are balanced on a rope which goes over a pulley. One remains stationary while the other climbs the rope. What happens?

23. Write 20, using only four 9's.

24. A watermelon weighed $\frac{9}{10}$ of its weight and $\frac{9}{10}$ of a pound. I ate a slice consisting of $\frac{1}{3}$ of the melon. What was its weight?

25. Can you distribute ten pieces of sugar in three cups so that every cup shall contain an odd number of pieces.

26. A glass is half full of wine and another twice the size is one third full. They are then filled with water and the contents mixed. What part of the mixture is wine?

JOHNSON'S CAT

+ 27. Johnson's cat went up a tree,
Which was sixty feet and three;
Every day she climbed eleven,
Every night she came down seven.
Tell me, if she did not drop,
When her paws would touch the top.

28. Given $[3(230 + l)]^2 = 492104$, find l by inspection, not knowing the value of l . Only simple arithmetic required.

29. With six matches form four equilateral triangles. The problem is to lay three matches on a table, forming an equilateral triangle; that is, a triangle having three equal sides. Then with the remaining three matches, placed in connection with these, form three other triangles exactly the same size as the first. Do not give it up too soon; it can be done.

30. How many cubic inches of dirt in a hole the dimensions of which are one foot; that is, a cubic foot? *none*

31. A tramp came to a merchant and purchased a pair of blankets for \$8.00, and gave the merchant a fifty-dollar bill. The merchant, not being able to change the bill, took it to the bank and got it changed. After the tramp had gone with his blankets and change, the banker called and informed the merchant that the bill was counterfeit, and he would have to make it good. After the merchant had made it good with the bank, what was the total loss of the merchant?

32. Express 8 by using eight eights, fractions not being permissible.

33. Express 3 by using three threes.

34. Eight men went to a hotel for rooms. They said that they wanted separate rooms. The proprietor informed them that he had only seven rooms, but would accommodate them all right and give each one a separate room. He accordingly placed two of the men in the first room, then he placed the third man in the second room, the fourth man in the third room, the fifth man in the fourth room, the sixth man in the fifth room, the seventh man in the sixth room. He now had one room, the seventh left, and he accordingly went to the first room where he had placed the two men, took one of them out and put him in room number seven; thus, as the example states, putting eight men in seven rooms and giving them all separate rooms.

35. If a brick balances with three quarters of a brick and three quarters of a pound, then how much does a brick weigh?

36. Find the cost of making six pieces of chain of five links each into an endless chain of thirty links if it costs a cent to open a link and costs a cent to weld it.

37. A was complaining to B that he agreed to pay \$80 cash and a fixed number of bushels of wheat as the yearly rental for his farm. That, he explained, would amount to just \$7 an acre when wheat was worth 75 cents a bushel, but as wheat was now worth \$1 a bushel he was paying \$8 per acre, which he thought was too much. Can you tell the size of the farm?

38. Tell how much is lost by measuring off 20 feet of rope with a yard measure 3 inches too short.

39. Two men were born at the same moment and died at the same instant, yet one was 100 days older than the other. How can this be accounted for?

40. How many times does a clock strike in a day if it only strikes the hours?

41. A is 10 years older than B, and the sum of their ages is 5 times their difference. What are A's and B's ages?

42. A boy asked a farmer how many pigs he had. "You shall reckon for yourself," said the farmer. "If I had as many more, and half as many more, and seven to boot, I should have 32." How many had he?

43. Another farmer being asked the same question replied, "If I had as many more, and half as many more, and two pigs and a half, I should have just a score." How many had he?

44. Take 45 from 45 and leave 45 as a remainder.

45. Can you arrange the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 and 0 so that they will add up to give just one (1)?

46. This and that and a half of this and that is what per cent of this and that?

47. "Figures won't lie even though liars may figure." Is the preceding statement true?

48. If you can multiply feet by feet and get square feet, can you not multiply dollars by dollars and get square dollars, or eggs by eggs and get square eggs?

49. A man having \$50 in a bank withdrew it as follows:

\$20	leaving	\$30
15	leaving	15
9	leaving	6
6	leaving	0
<u>\$50</u>		<u>\$51</u>

Now where does the extra dollar come from?

50. If there are more than 366 persons in a room, how can you be certain that at least two of them have the same birthday?

51. Use the ten figures 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0 without duplication and write a number whose sum is 100.

52. A customer goes into a drug store, orders a five-cent glass of Coca Cola and puts down a dollar bill in payment. The druggist says, "I cannot change it." Whereupon the customer replied, "I'm sorry, but the only other money I have is a five-dollar bill."

"That's all right," declared the druggist, and hands the customer \$4.95 in change.

If he could not change the one-dollar bill, how could he change the five?

53. Thrice what number is twice that number?

54. From a number consisting of the nine digits subtract a number consisting of the nine digits and giving a remainder consisting of the nine digits.

55. From a number consisting of the ten digits subtract a number consisting of the ten digits and giving a remainder consisting of the ten digits.

56. Write a number consisting of the nine digits. Multiply it by 8 and obtain a result consisting of the nine digits.

57. Does 11 and 2 make 1?

58. A fraction may be expressed thus, using six pins.



Make the fraction express an integer simply by moving a single pin.

59. A certain number was divided by one more than itself, giving a quotient of one fourth. Another number was divided by one more than itself, giving a quotient of one fourth of itself. Find the product of the numbers?

60. I sold a house for \$4000. I then bought it back for \$3500 and then sold it for \$4500. How much did I gain?

61. Arrange the figures 1, 2, 3, 4, 5, 7, 8, and 9 in two groups of four figures each so that the sum of one group will be equal to the sum of the other group. You are not permitted to reverse the nine to form the missing six.

62. A postmaster said, "I will give you 13 stamps for a cent and a quarter." What was the cost of a stamp? $2\frac{1}{4}$

63. "Write down," said a professor to his senior class, "the nine digits in such an order that the first three shall be one third of the last three and the central three the result of subtracting the first three from the last." Can you discover the arrangement?

64. What would 10 be if 4 were 6?

65. Two coins *A* and *B* of the same size are placed upon a table so that *A* is tangent to *B*. If *B* is kept fixed and *A* is rolled around *B*, always remaining tangent to *B*, how many revolutions does *A* make in rolling once around *B*?

Answer before experimenting.

66. Can you place 4 coins on a table equally distant apart?

67. Can you place 4 marbles equidistant apart?

68. If one third of six apples sell for three cents, what will one fourth of forty apples sell for?

69. Sweet milk sells for three cents a pint, and sour milk sells for three pints for a cent. A mother gives her son one cent and two pails, telling him to bring home an equal quantity of each kind of milk, but that the one cent is to cover exactly the total cost. How much of each can he bring home?

+ 70. As I was traveling on the forest grounds
Up starts a hare before my two greyhounds:
The dogs being light on foot they fairly run
Unto her 16 rods just 21,
The distance she started up before
Was fourscore sixteen rods just, and no more.
Now, I'd have you unto me declare
How far they ran before they caught the hare?

71. Eggs, as many more eggs, half as many eggs are 2 more than twice as many eggs less 2. How many eggs?

72. Being out of money I asked my friends A, B, and C if one of them could not lend me \$10. A replied, "Just twice what I have less \$14"; B replied, "Twice mine less \$16"; and C replied, "Thrice mine less \$35." All being willing to lend, I decided to borrow from each just enough to enable each of us to have \$10. How much did I borrow from each?

+ 73. ^{4 2 3 7 5} HOW MANY FISH DID I CATCH?

One half I caught less two fell back;
One sixth plus six are in the sack;
One half the remainder less three I fried;
One third less five a turtle tried;
One half in sack, though very few,
If you will cook I'll lunch with you.

NUTS FOR THE FIRESIDE

✓ 1. A watermelon weighs $\frac{9}{10}$ of its weight and $\frac{9}{10}$ of a pound. How many pounds does it weigh?

2. Which is correct to say, 6 and 7 are 14, or 6 and 7 is 14?

3. Which would you prefer an old \$10 bill or a new one?

4. How much dirt is there in a hole the dimensions of which are an inch?

5. Write 2 using only seven 2's.

6. What is a third and a half of a third of 4?

7. $3 + 3 \times 3 - 3 \div 3 - 3 = ?$

8. I bought a goose that weighed 10 pounds and a half of its weight besides. What was its weight?

9. If 6 cats eat 6 rats in 6 min., how many cats will it take to eat 100 rats in 100 minutes at the same rate? 99

✓ 10. If an apple balances with $\frac{3}{4}$ of an apple of the same weight and $\frac{3}{4}$ of an ounce, how much does the apple weigh?

11. How long would it take you to cut a 60-yard piece of cloth into 1-yard lengths, at 1 minute to each cut?



12. What will it cost to make the above five pieces of chain of three links each into one straight chain, if it costs 2 cents to cut a link and 2 cents to weld a link?

13. What part of $\frac{1}{2}$ square yard is $\frac{1}{2}$ yard square?

14. How many quarter-inch squares will it take to make an inch square, and how many quarter-inch cubes to make an inch cube?
- ✓15. A bottle and a cork cost \$1.10 and the bottle costs \$1.00 more than the cork. How much did each cost?
16. A hunter walked around a tree to kill a squirrel; the squirrel kept behind the tree from the hunter. Did he go around the squirrel?
17. Do you know your A, B, C's? What is the middle letter?
18. Which, if either, is the greater, six dozen dozen or a half a dozen dozen?
19. A man has a wolf, a goat, and a cabbage to carry across a river, one by one, on account of the smallness of the boat. In what manner can this be done, that the wolf may not be left with the goat nor the goat with the cabbage?
20. In a box are six oranges. It is required to divide these among six boys, in such a way as to leave one orange in the box. How can this be done without cutting the oranges?
21. There is a room with eight corners and a cat in each corner, seven cats before each cat and a cat on every cat's tail. How many cats are there in the room?
22. Two ducks before a duck, two ducks behind a duck, and a duck in the middle, are how many ducks?
23. Twenty-one ears of corn are in a hollow stump. How long will it take a squirrel to carry them all out if he carries out 3 ears a day?
24. Two men laid a wager as to which could eat the more oysters; one ate ninety-nine, and the other a hundred and won. How many did both together eat?
25. After killing a certain number of cattle, it was found that twenty fore feet remained. How many head were killed?

26. A man had twenty sick sheep and six of them died. How many did he have left?
27. Write an even number using only figures representing odd numbers.
28. Write one hundred in our common numerals without using any zeros.
29. A snail crawling up a pole 10 ft. high climbs up 3 ft. each day and slips back 2 ft. each night. How long will it take the snail to reach the top?
30. If a herring and a half costs a cent and a half, how much will a dozen and a half herring cost?
31. A boy went to a spring with a 5 and a 3 quart measure to procure exactly 4 quarts of water. How did he measure it?
32. What is the difference between twice twenty-five and twice five and twenty?
33. Can you write 30 with 3 equal figures?
34. On a limb of a tree sat 9 birds. A boy shot 3 of them. How many remained?
- ✓35. After cutting off 10% of a piece of cloth a merchant had 100 yards left. How much had he at first?
36. Which is heavier, a pound of gold or a pound of feathers? A pound of lead or a pound of feathers?
37. A lady with \$1 wanted \$1.25. She pawned the \$1 for 75 cents and then sold the pawn ticket for 50 cents. She then had the \$1.25. Who lost on the transaction?
- ✓38. Two shepherds were feeding their flocks on the mountain-side. Said one to the other: "Jack, give me one of your sheep, and I shall have as many as you." "Nay," replied the other greedily. "Give me one of yours and I shall have as many again as you." How many sheep had each?
39. What is the difference between twenty four-quart bottles and four and twenty quart bottles?

40. Show that six and six are eleven.
41. Show that half of 888 is twice as much as nothing.
42. What number is the same when inverted or turned, wrong side up?
43. Show that four and four are nine.
44. Can you take 1 from 19 and get 20.
45. Add 1 to 9 and make it 20.
46. Write the number twelve thousand twelve hundred twelve.
47. A cow standing on 3 feet weighs 300 lb. How much will she weigh standing on her 4 feet?
48. Three clowns are standing in a row wearing the figures 1, 6, and 8 respectively. How would you arrange them so that the three-figure number formed shall be exactly divisible by 9?
49. I bought a great big pumpkin that weighed 10 pounds, half of 10 pounds, and half of its own weight besides. What did the pumpkin weigh?
50. What is that number which, divided by 2, will leave 1?
51. Write 13 in two ways, using only 3 x 's.
52. Write 17 in two ways, using only 4 x 's.
53. Write 19 in two ways, using only 5 x 's.
54. Solve examples 51, 52, and 53 for x .
55. Find a proper fraction which equals its reciprocal.
56. If it takes 12 one-cent stamps to make a dozen, how many two-cent ones does it take?
57. Can you add 1, 2, 3, 4, 5, 6, and 7 in rotation and total 100?
58. A girl divided a certain number into four parts. She then added 2 to the first, subtracted 2 from the second, multiplied the third by 2, and divided the fourth by 2. She was

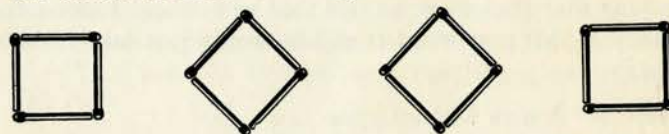
astonished, as the result was the same in each case. How is this?

59. Suppose some one would offer you 21 pigs for your very own, providing you could build four pens for the 21 young porkers so that each pen would contain an even number of pairs and an odd pig besides. Could you do it?

60. Find two fractions such that their difference is equal to their product.

61. Eight boys were wearing the numbers 1, 2, 3, 4, 5, 7, 8, 9 respectively on their backs. They divided themselves in two groups. Those bearing the numbers 1, 2, 3, 4 constituted one group and those wearing the numbers 5, 7, 8, 9 the other. It was observed that the sum of the numbers in the first group was 10 while the sum of the numbers in the second was 29. Can you rearrange the boys in two new groups so that the sum of the numbers in each group will be the same?

62. Given 16 matches arranged as follows:



Take up four matches and move one so as to spell what matches are made of.

63. Show how to arrange six matches so that each match shall touch four others.

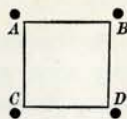
64. My money in \$2.00 bills is

$$[2 + 2 \times 2 - 2 \times 2]^{2-2} - 2 \div 2.$$

How much?

65. If I had as many more dollars, twice as many more dollars, and one half as many dollars less \$9.00, I would have nothing. How much have I?

66. A camper pitched a square tent between four trees, A , B , C , and D , forming a square. After his departure another camper came along and also pitched a square tent with twice the floor space between the same trees? How was it done?



67. A man has a square plot of ground with 12 posts equally spaced on each side. What is the total number of posts?

68. Three brothers were left an estate which was to be divided as follows: the oldest was to receive $\frac{1}{2}$, the next $\frac{1}{3}$, and the next $\frac{1}{6}$. In the estate was 17 goats. The boys wished to divide the goats among themselves without having to sell any. They were discussing how this could be done when a neighbor offered to solve the problem for them by placing his goat with the 17 which would make 18. The oldest then would receive 9, the next 6 and the next 2. The neighbor asked them if they were satisfied and on their replying that they were, he said that as 9, 6 and 2 amounted to only 17, his goat was left and he would just take it back. Explain.

69. A man without eyes
Went out to view the skies
He saw an apple tree with apples on it
He neither took apples nor left apples
How many apples were on the tree?

70. Thrice as many ducks are 3 more than twice as many. How many ducks?

71. If yours and mine were ours, ours would be thrice yours and twice mine. What have we?

72. If you divide by 5 or subtract 5, the result is the same. For what number is this true?

NUTS FOR THE CLASSROOM

WALKING ON THE EQUATOR

1. If a man 6 feet high could walk around the earth on the equator, how much farther would the top of his head move than his feet?

THE EARTH'S EQUATOR

2. Suppose 48 inches be added to the earth's equator. What change will be made in the earth's radius? (Make an estimate mentally and then solve.)

WHAT IS GRANDPA'S AGE?

3. A little girl said: "I am five years old, grandpa. How old are you?" The old gentleman replied: "Your father is 11 times as old as you are, and I am as many years old as he will be when you are one third my age. Do you know how old I am now?" Can you help the little girl out?

AN EGG PROBLEM

4. If six dozen eggs cost as many cents as I can buy eggs for 50 cents, what are eggs worth per dozen?

COST OF DRAWING A BILL

5. In a bag are 3 five-dollar bills and a ten-dollar bill. How much should be paid for permission to draw a bill from it?

FIND THE GOOSE'S WEIGHT

6. A goose weighs $\frac{4}{5}$ of its weight and $\frac{4}{5}$ of a pound. What is the weight of the goose?

WHAT DOES THE ORANGE WEIGH?

7. An orange just balances in weight with $\frac{3}{4}$ of an orange of the same weight and $\frac{3}{4}$ of an ounce. What is the weight of this orange?

HOW FIND THE FIGURE?

8. Write any number of as many figures as you please and I will name a single figure that may be placed before, after, or anywhere within the number making it exactly divisible by 9. How do I find the figure?

WIFE TO HUBBIE

9. Wife to Hubbie: "When you are 15 times the age I was when you were half as old as I am, you will be one half again as old as I would be were I as much older than you are as you are older than I am." Says Hubbie to Wife: "When you will be as much older than you are as you are younger than I was year before last, our combined ages will be 50." What are their ages?

HOW MANY POSTS?

10. A farmer having a garden 10 rods square wishes to know just how many posts will be required to inclose his land. The posts are to be exactly a rod apart. How many will be required?

FIND THE AGES OF A, B, AND C

11. Reversing the digits of A's age gives B's age. The difference between A's age and B's age is twice C's age. B's age is 10 times C's age. Required the age of each.

ASKING FOR A MAN'S DAUGHTER

12. When a young man asked an old man for the hand of his daughter in marriage the latter replied: "If you will go into the cellar and bring back enough apples to give me half that you bring back, and half an apple over, and then give the girl's mother half that you have left and half an apple over, then give the girl half what you have left and half an apple over and have just one apple left, without having to cut any apple you may have her." How many apples would the young man have to bring?

A DOG AND GOOSE CHASE

13. A dog is in the center of a circular pond 200 feet in diameter and a goose is swimming around the outer edge of the pond. The dog starts toward the goose, swimming at the same rate the goose is swimming. He continually keeps in line with the center of the pond and the goose. How far will he swim before reaching the goose?

FIND THE SHARE OF EACH?

14. Ann and Mary contracted to dig a patch of potatoes for \$10. Ann can pull tops as fast as Mary can dig and Mary can pull tops three times as fast as Ann can dig. How should the \$10 be divided between them so that each will get her just share?

DIVIDING PEANUTS

15. A woman with four children bought a sack of peanuts. To the oldest child, a boy, she gave one peanut and $\frac{1}{4}$ of what remained, and likewise to each of the other children. The second was a girl, the next a boy, and the last a girl. It was found that the boys had received 100 more nuts than the girls. How many did the mother keep for herself?

A FAMILY OF FIFTEEN CHILDREN

16. In a family were 15 children, each one one and one half years older than his next brother or sister. How old is each child if we know that the sum of three of their ages — the 9th, 11th, and 15th — equals the age of the first?

ANTECEDENT AND CONSEQUENT

+ 17. The antecedent of a certain ratio is 10. The inverse ratio is nine times the direct ratio. By what must the consequent be multiplied that the direct ratio may be nine times the inverse ratio?

A CLOCK PROBLEM

18. A boy noted the position of the hands of a clock and went for a swim. Between two and three hours later he came back and found that the hands had exchanged places. How much did the swim exceed the two-hour limit set by the boy's mother?

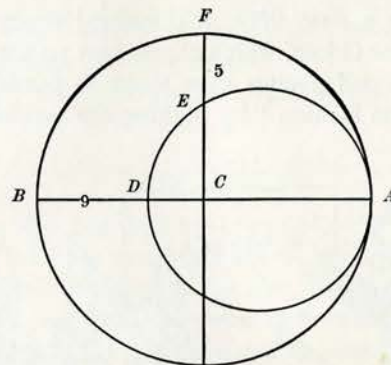
HOW MANY EGGS IN THE BASKET?

19. A market-woman, when asked how many eggs she had in her basket, replied that she didn't know, but when she put them in by twos, there was one left over, and when by threes, by fours, by fives, or by sixes, there was one left over; but when she put them in by sevens, they came out even. How many eggs were there in the basket?

THE FOUR TRAVELERS

20. Four commercial travelers leave the home office on Thursday, January 2, 1923, on circuits of 4, 8, 12, and 16 weeks' travel respectively. If they continue traveling every week, repeating their circuits, on what date will all first meet again at the office?

FOR THE GEOMETRY CLASS



21. C is the center of the large circle. $BD = 9$ and $EF = 5$. Find the diameters of the circles.

HOW FAR FROM WICHITA FALLS TO DALLAS?

22. Bill left Wichita Falls for Dallas simultaneously with the departure of Phil from Dallas. They met and exchanged the fraternal grip at a point where Bill had gone 18 miles farther than Phil. After an affectionate parting, it took Bill $13\frac{1}{2}$ hours to reach Dallas and Phil 24 hours to reach Wichita Falls. How far was it from Wichita Falls to Dallas?

AN AUTO AND ITS TIRES

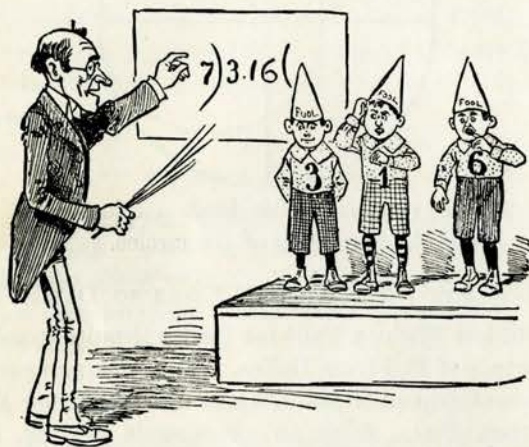
23. An auto is twice as old as its tires were when the auto was as old as the tires are. When the tires are as old as the auto is, the combined ages of the tires and auto will be two years and three months. How old is the auto?

HOW OLD ARE JOHN AND MARY?

24. John is thrice as old as Mary was when he was as old as she is now. When she is as old as he is now, the sum of their ages will be 100. Find their ages now?

SAM LOYD'S STUDY IN DIVISION

- + 25. Here is a cute little arithmetical study, told in the style of Mother Goose, which shows how in the days of auld lang syne the pedagogues were wont to punish the dunces in true Mikado fashion "by making the punishment fit the crime."



The sketch shows three little dunces who failed ignominiously to do that simple sum in division on the blackboard, so, as the story says:

"Those stupid boys, who were so dumb,
They could not do a simple sum,
Were marked with numbers three, one, six,
And told those numbers they could mix,
And find by many changes tried
A sum which seven would divide!
You will say the answer is so plain
That all who fail, dunce caps should gain!"

CATS AND KITTENS

26. 4 cats and 3 kittens weigh but 37 pounds while 3 cats and 4 kittens weigh but 33 pounds.

What does a cat and a kitten weigh? A cat? A kitten?

A FARM QUESTION

27. A man who has only two rows of corn hires A and B to hoe them. A hoes three hills on B's row and then begins on his own row. B finishes his row and hoes six hills on A's row, when they find the work is finished. Which man hoes the more and how much more, the rows containing the same number of hills?

AN OUT-OF-DATE PROBLEM

28. A man hires a livery team to drive from A to C via B and return for \$3.00. At B, midway between A and C, he takes a passenger to C and back to B. What should he charge the passenger?

A FARM PROBLEM

- + 29. Some horses and chickens are in a barn; the total number of heads and wings equals the number of feet. How many horses and how many chickens are there?

A GOOD ONE

- + 30. If you walk halfway to the door and then walk half of the remaining distance, and then half of what is left, and so on continually, how long will it take you to get out of the room?

MAKE YOUR GUESS

- + 31. Below are given two columns consisting of the same figures, but in reversed order.

(a)	(b)
1 2 3 4 5 6 7 8 9	1
1 2 3 4 5 6 7 8	2 1
1 2 3 4 5 6 7	3 2 1
1 2 3 4 5 6	4 3 2 1
1 2 3 4 5	5 4 3 2 1
1 2 3 4	6 5 4 3 2 1
1 2 3	7 6 5 4 3 2 1
1 2	8 7 6 5 4 3 2 1
1	9 8 7 6 5 4 3 2 1

Which column when added will give the larger result? Make your guess. Then test the correctness of your guess by adding.

WHAT IS MY AGE?

- + 32. "Five times seven and seven times three
Add to my age and it will be
As far above six nines and four
As twice my years exceeds a score."

AN AUTO RACE

33. A and B are racing on a circular track. A is 24 miles from the starting point. He is twice as far as B was, when A was as far as B is now, since which time they have traveled at equal rates of speed. How far has "B" gone?

FIND THE AVERAGE RATE

34. I travel at the rate of 20 miles per hour in going to a place and 30 miles per hour in returning. What was the average rate in traveling?

A MISSING FIGURE

35. A teacher read a number to his class as follows:

123454638459854321

and said divide by 9. After giving them time to finish he called for the answer. Various answers were given. He then exclaimed, "No one has the correct answer for there is no fraction in the answer." In reading the problem again it was found that a figure was omitted. Find the missing figure.

HUNTING TURKEYS

36. A man killed 20 turkeys while hunting. He fired 8 times and killed an odd number each time. How many did he kill each shot?

THE RESULT ALWAYS THE SAME

37. Write any number of four figures in descending order. Reverse the order of the digits and subtract. Again reverse the order of the digits and add. The sum is the same for any number selected.

For example, select the number 9531. Reversing the digits we have 1359. We then have

9531 — The number selected.

1359 — Reversing the digits.

8172 — The difference.

2718 — Reversing the digits.

10890 — The sum.

FIND THE COST

38. I bought a broncho down in Texas for \$26, and after paying for his keep for a while sold him for \$60. That looked like a profitable deal. Nevertheless, I found that I had lost just half of the original price and one quarter of the cost of keep. Can you figure out just how much I lost on the deal?

FOR THE ARITHMETIC CLASS

39. What is the smallest number which, divided by 2, will give a remainder of 1; divided by 3, a remainder of 2; divided by 4, a remainder of 3; divided by 5, a remainder of 4; divided by 6, a remainder of 5; divided by 7, a remainder of 6; divided by 8, a remainder of 7; divided by 9, a remainder of 8; divided by 10, a remainder of 9?

FOR THE BRIGHT STUDENT

40. A man is twice as old as his wife was when he was as old as she is now. When she is as old as he is now, the sum of their ages will be 100 years. Find their ages now.

THE 1928 PRESIDENTIAL ELECTION

41. Can you read the following?

Hoover	1
Curtis	1 2
Smith	0 2 1
Robinson	0 2 1 2

A FENCE PROBLEM

42. Jones and Brown, two bargain hunters, were in consultation with Smith the fence builder.

"My property is exactly a square mile," said Jones.

"Mine is exactly a mile square," said Brown.

"Then there is no difference," said Smith. "I will fence either piece of property for \$1000." What did Smith charge per mile for fencing?

A FIGURE PUZZLE

43. Determine the sum of the numbers from 1 to 50 inclusive without adding them.

A CLOCK PROBLEM

44. A teacher in talking to his student asked: "John, how long did it take you to solve that problem?" He replied, "I did it between 4 and 5 o'clock yesterday evening. When I began, the hands of the clock were together, and when I finished, they were opposite each other." The teacher congratulated him upon his answer and said, "But what I want to know is, how many minutes you spent in doing the work."

A LADY TO HER LOVER

+ 45. You, sir, I ask, to plant a grove
To show that I'm your lady love.
This grove though small must be composed
Of twenty-five trees in twelve straight rows.
In each row five trees you must place
Or you shall never see my face.

THE THREE JEALOUS HUSBANDS

46. Three jealous husbands, traveling with their wives, find it necessary to cross a stream in a boat which holds only two persons. Each of the husbands has a great objection to his wife crossing with either of the other male members of the party unless he himself is also present. How is the passage to be arranged?

AN OLD PROBLEM

47. Two boys sit down to eat, one with 5 cakes and the other with 3, all the cakes having the same value. A third boy comes along and eats with them, paying 8¢ for his part of the meal. If they eat equally and consume all the cakes, how should the 8¢ be divided?

WHAT DID THE MELON WEIGH?

48. A melon placed in one of the scales of a false balance was found to weigh 16 pounds. When placed in the opposite scale, it weighed only 9 pounds. What was the correct weight of the melon?

CAN YOU FIND IT?

49. Discover a number that will be doubled by adding 24 to $\frac{2}{3}$ of it.

THE BOY'S ERROR

50. A boy divided a number by 6 instead of multiplying, obtaining 15 as a result. Find the true answer.

AN INCORRECT YARD STICK

51. I make 20% instead of 40% on goods, as my yard stick is too long. Find its length in inches.

A FLY PROBLEM

52. A fly takes the shortest route from a lower to the opposite upper corner of a room 18 feet long, 16 feet wide, and 8 feet high. Find the distance it walks.

A RACE TRACK PROBLEM

53. If a race track is 5 yards wide and a horse be π more seconds in traveling his best on the outer than inner edge, what is his rate per second?

REVERSING THE DIGITS

54. If we reverse the two digits of a number and subtract this from the number itself, we get 27. Find the difference between the digits.

ANOTHER AUTO RACE

55. Two autos A and B start from the points C and D opposite each other across a lake. They start towards each other and maintain uniform speeds. They meet 421 feet from D, but continue their speed and meet again 221 ft. from C. What is the diameter of the lake?

CALVIN AND HERBERT

56. Calvin is $\frac{3}{4}$ times as old as Herbert was when Calvin was 10 years older than Herbert is now. The sum of their ages is 100 years. Find their ages.

HOOVER'S INAUGURATION — MAR. 4, 1929

57. Two brothers, Bill and Tom, on that day were trying to determine their ages.

Bill to Tom, "I am wondering just how old we are. I well remember that your age was thrice mine when Taft was inaugurated President."

Tom to Bill, "I can't recall either of our ages or our dates of birth, but I remember very distinctly that when Harding was inaugurated, my age was twice yours." Can you reveal unto them their ages?

WHAT WERE OUR AGES AT SISTER'S DEATH?

58. My age was thrice yours when sister was born, but twice yours at her death. Had she lived 10 years longer the sum of our three ages would have been 100.

What were our ages at her death?

A BRAIN TEASER

59. Required, the smallest number which, being divided by 2, 3, 4, 5, 6, 7, 8, 9, will leave 1, 2, 3, 4, 5, 6, 7, 8.

MATHEMATICAL NUTS

A GRAZING PROBLEM

- + 60. In the midst of a meadow, well covered with grass,
Just an acre was needed to tether an ass;
How long was a line that reaching all 'round,
Restricted its grazing to an acre of ground?

THE TWO DAUGHTERS

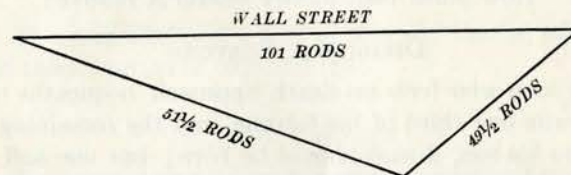
61. A wealthy man two daughters had,
And both were very fair;
To each he gave a tract of land,
One round, the other square.
At twenty shillings an acre just
Each piece its value had;
The shillings that did encompass each
For it exactly paid.
If 'cross a shilling be an inch,
Which had the greater fortune,
She that had the round or square?

A BIT OF GEOMETRY

62. I place a bowl into the storm,
To catch the drops of rain;
A half a globe was just its form,
Two feet across the same.
The storm was o'er, the tempest past,
I to the bowl repaired;
Six inches deep the water stood,
It being measured fair.
Suppose a cylinder, whose base
Two feet across within,
Had stood exactly in that place:
What would the depth have been?

NUTS FOR THE MATH CLUB

MY NEW YORK PROPERTY



1. I own the above lot located in the wealthiest section of New York City and am desirous of knowing its exact area. The dimensions according to the deed are exactly 101 rods, $51\frac{1}{2}$ rods and $49\frac{1}{2}$ rods respectively.

I guarantee to have just what the deed calls for and wishing to share my wealth with friends have decided to reward each of the first 99 sending in correct solutions showing the exact area of the tract with a .01 interest in the land. You are only asked to pay the cost of the deed. What is the area?

A GOOD CHAIN PROBLEM

2. A miner asks for room and board at a hotel for fifty days. He is told that the price is \$1 a day. He has no money, but he has a heavy gold chain with fifty links, each worth a dollar. He agrees with the landlord to cut the chain and pay one link each day. At the end of fifty days the miner will redeem the chain and have it soldered together.

The miner does not wish to cut the chain any more than is necessary. How many links will it be absolutely necessary for him to cut?

The question of making change with the links enters into the problem very strongly. The answer is the smallest number of links that will be necessary for him to sever.

THE HORSE TRADERS

3. A and B propose to trade horses. A asks "\$20 to boot," and B asks \$10. They finally agree to split the difference. How much boot money should A receive?

DIVIDING A FORTUNE

4. A man who feels his death approach bequeaths to his young wife one third of his fortune, and the remaining two thirds to his son, if such should be born; but one half of it to the widow and the other half to his daughter, if such should be born. After his death twins are born, a son and a daughter. How should the fortune be divided amongst the three?

A WATCH PROBLEM

+ 5. A teacher looks at his watch when leaving school for dinner. When he returns, he finds that the hour hand and minute hand have just changed places (the positions they had when he left the school). What time was it when he left, and what time when he came back to school?

HANS, KLANS, AND HENDRICKS

6. Three Dutchmen and their wives went to market to buy hogs. The names of the men were Hans, Klans, and Hendricks, and of the women, Gertrude, Anna, and Katrine; but it was not known which was the wife of each man. They each bought as many hogs as each man or woman paid shillings for each hog, and each man spent three guineas more than his wife. Hendricks bought 23 hogs more than Gertrude, and Klans bought 11 more than Katrine. What was the name of each man's wife?

TO FIND THE SPEED OF A TRAIN

7. For how many seconds must I count the clicking of the rails under a train that the number of rails counted may be equal to the speed of the train in miles per hour, a rail being 30 ft. long?

THIS AND THAT

8. This and that and a half of this and that is what per cent of three fourths of this and that?

A LAND PROBLEM

9. Two men, A and B, buy a section of land and pay \$320 each. After purchasing the land they find that the section is composed of two grades of land and decide to dissolve partnership. One kind is worth \$.75 per acre and the other is worth \$1.25 an acre. How many acres did each man get, provided A takes the land worth \$.75 per acre and B the land worth \$1.25 per acre?

WHAT JOHNNIE TOLD

10. "Sister's age is just three times mine," replied Johnnie to Charlie Slowpop, who elicited some family history from "her" young brother. Then the boy continued, "I was 40 months old when we moved to this town, Father is now $1\frac{1}{3}$ times as old as he was then, and I am two years more than half as old as Mother was when we came here. When I am as old as Father was when we came to town, the combined ages of Father, Mother, and me will amount to just 100 years." "That will be about all for the present," remarked Charlie, who heard a footfall on the stairs.

Can you tell from Johnnie's confidential talk just how old he was at the time he gave the facts?

MAKING CHANGE

11. I entered a fruit stand with a friend and bought fruit amounting to exactly 34 cents. When I went to pay, I found that I did not have the change. I only had a dollar, a 3-cent piece, and a 2-cent piece. The merchant could not make the change. He only had a 50-cent piece and a 25-cent piece. My friend wishing to assist us offered to lend us all he had. He had 2 dimes, 1 nickel, a 2-cent piece, and a 1-cent piece. I was thus enabled to pay for the fruit and in addition each one received what was due him in full. Show how the transaction was made.

THE THREE MISSIONARIES AND THREE CANNIBALS

12. Three missionaries and three cannibals are desirous of crossing a river. They can't swim, but find a boat which will carry only two. It is also found that each of the missionaries can row the boat but only one of the cannibals can do so. A missionary is not afraid to be in the presence of one cannibal, but for one missionary to be left in the presence of two cannibals would mean that the missionary would be instantly eaten by the cannibals. Now the problem is to safely deliver the six on the other side of the river.

AN OLD TIMER

13. If three rails will fence three acres, what will four rails fence?

THE SPIDER AND THE FLY

+ 14. A room is 30 feet long, 12 feet wide, and 12 feet high. A spider starts from a point in the middle of the end wall of the room one foot from the floor and crawls to a fly in the middle of the opposite wall one foot from the ceiling. How far does he travel if he takes the shortest path?

A WATCH PROBLEM

+ 15. A jeweler repairing a watch placed the hour-hand upon the minute-hand pinion, and the minute-hand upon the hour-hand pinion and then set the watch at six o'clock, which was the correct time. He then gave the watch to the owner. The latter sometime later wishing to know the time of day, and not knowing that the hands had been changed, looked at his watch and found that it registered the correct time. What was the time of day?

OTHER CHAIN PROBLEMS

16. A man owning a valuable gold chain, but having no money, wanted to engage board at a lodging house. An arrangement was made with the proprietor to accept one link of the chain for each day's board. The chain consisted of 63 links. As the owner hoped to redeem the chain at some future date, he wanted to cut as few links as possible. How many links did he find it necessary to cut in order to pay for 63 days' board, giving the proprietor one link each day?

17. Solve the above problem for 7 links and 7 days.

18. Solve it for 23 links and 23 days.

19. Develop a formula for finding the maximum number of links in chains to be used as in problems 16-18 inclusive where the number of cuts are known.

FIND THE AGE OF EACH MEMBER OF THE FAMILY

20. When I was born my sister was one fourth as old as my mother. She is now one third as old as my father, and my own age is one fourth of my mother's. In four years I shall be one fourth as old as my father. How old am I, my parents, and my sister?

THE ERASED PROBLEM

21. A boy after solving and erasing a problem in long division was asked to replace his work. He couldn't recall either the dividend, divisor, or quotient, but remembered that the successive subtrahends were 690, 2415, and 2070 and that there was a remainder of one. He promptly replaced the problem. Can you do likewise?

AN ENGLISH MONEY PROBLEM

22. Name the largest and smallest sums of money that can be written in pounds, shillings, pence, and farthings, using each of the nine digits once and only once.

AN ALABAMA FARMER'S ANIMALS

23. An Alabama farmer had 5 pastures. On each of these pastures grazed the same number of animals consisting of cows, hogs, and sheep. He sold all his stock to eight dealers. Each dealer bought the same number of animals, paying \$17 for each cow, \$4 for each hog, and \$2 for each sheep. The farmer received in all \$301. What was the greatest number of animals he could have had and how many of each kind did he have?

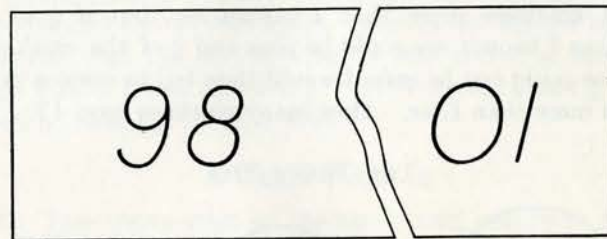
DIVIDING BY NINE

24. Write any integer of three or more figures. Divide by 9 and give me the remainder. Erase one digit of the number; divide by 9, and tell me the remainder. I will then promptly tell the figure that was erased.

THE TORN CARD

25. I was handed a card upon each side of which was printed in large characters a number of four figures, no two

of which were alike. The number on one side was 9801. Supposing the card to be of no value I tore it into two pieces as illustrated, but being somewhat of a mathematician



I observed that if 98 and 01 be added and the sum squared that the result was exactly 9801. I also found on the other side that the square of the sum of the parts gave exactly that number. Can you tell the number on the back of the card?

A FISH STORY

26. I went fishing and caught fish. "How many did you catch," I was asked. I replied, "You can divide my fish into two parts, so the difference between the two numbers is the same as the difference between their squares." Now the question is, just how many fish did I catch?

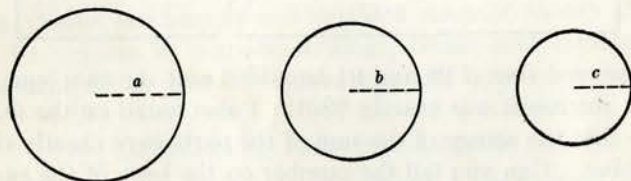
A PARTNERSHIP PROBLEM

27. A and B form a partnership in raising hogs. A furnishes the hogs and half the corn. B feeds them and furnishes the other half of the corn. B being out of corn one day feeds them 60 ears from A's crib. The next day they receive from their farm a load of corn in which A owns a third interest. How many ears should be returned to A from this load to make them equal?

HOW MANY CHICKENS HAVE I?

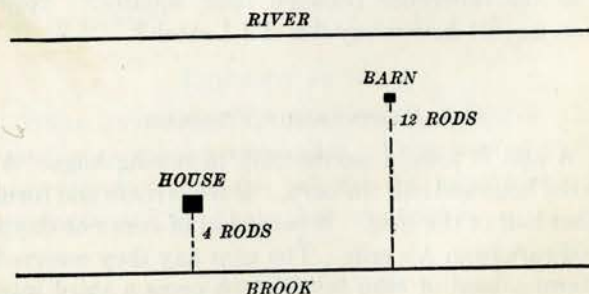
28. If $\frac{1}{2}$ the chickens I can see could not be seen and $\frac{2}{3}$ of those I cannot see, could be seen, I would then see six dozen chickens more than I cannot see, but if $\frac{1}{2}$ of the chickens I cannot see could be seen and $\frac{3}{4}$ of the chickens I can see could not be seen, I would then fail to see $\frac{1}{2}$ a dozen dozen more than I see. How many chickens have I?

THE THREE PIES



29. Four boys are given three pies of different radii a , b , and c , but equal in thickness. They wish to divide them equally among them by making as few pieces as possible. Can you make the division, knowing that $a^2 = b^2 + c^2$.

FROM THE HOUSE TO THE BARN



30. Mr. Jones in going from his house to his barn carried two buckets. He went to the brook and filled one bucket;

proceeded to the river and filled the second; and then proceeded for the barn.

"How far did you go?" Jones was asked. He replied, "I do not know, but remember that the river and brook were parallel and 20 rods apart. Also in leaving the brook I observed that the line in which I was traveling made an angle with it whose sine was $\frac{4}{5}$." How far did Jones go, provided he took the shortest route?

A TRAIN PROBLEM

31. Two trains start at the same time, one from Dallas to Houston, the other from Houston to Dallas. If they arrive at their destinations one hour and four hours respectively after passing one another, how much faster is one train running than the other?

+ TIME PROBLEMS

32. If you add one quarter of the time from noon till now to half the time from now till noon to-morrow, you will have the exact time. What is the time?

33. How many minutes is it until six o'clock if 50 minutes ago it was 4 times as many minutes past three o'clock?

34. The minute and hour hands of my watch are together every 65 minutes. Does it gain or lose time and how much?

TOSSING COINS

35. If five quarters are tossed at the same time, what are the chances that at least four of them will turn up all heads or all tails?

A LAND PROBLEM

36. Four men buy a circular piece of land and wish to divide it into four similar tracts, each tract to be bounded by arcs.

ANOTHER LAND PROBLEM

37. A father wills a circular tract of land of 1000 acres to his four sons. On this tract are located two fine artesian wells; one on the extreme eastern point of the boundary line and the other on the extreme western point. The father requests that each son must have the same amount of land and also that each must have access to each well. In making the division it was necessary to call in a mathematician. Can you make the division?

A SEE SAW PROBLEM

38. A boy having no playmate and desiring to play see saw tied a number of brick to one end of the plank to balance his weight at the other end. He found that he could balance against 16 brick on the short end or 11 brick on the long end. Find the boy's weight if a brick weighs three quarters of a brick and three quarters of a pound?

A FARM PROBLEM

39. If it requires 11 rounds of the binder to cut one half of a rectangular wheat field and 14 more to cut the remainder, find the ratio of the length of the field to its width. All swaths are full width.

A PROBLEM IN BASEBALL

+ 40. Two baseball teams, presumably of equal merit, are playing a series of games with the understanding that the first team winning four games wins the series. They have played two games, both won by A. What are the odds in favor of A's winning the series?

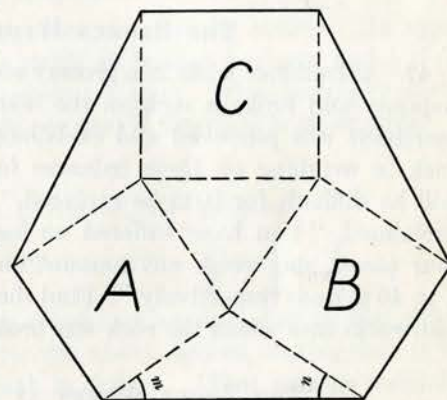
DIVIDING THE SACK OF MARBLES

41. Three boys were given a sack of marbles and it was agreed that they should be divided in proportion to their ages, the sum of which was $17\frac{1}{2}$ years. The sack contained 770 marbles and as often as John took 4 James took 3, and as often as John took 6 Sam took 7. How many marbles did each boy receive and what were their respective ages?

TWO VERY INTERESTING PROBLEMS

42. A, B, and C are squares. Given area of B 4.73 acres, area of C 5.27 acres, and $\angle m + \angle n = 135^\circ$, find the area of the entire figure.

43. Given areas of A, B, and C, 18, 20, and 26 acres respectively. Required the area of the entire figure.



THE NINE PIECES OF CHAIN



44. A man has nine pieces of chain, as shown in the illustration. He wishes to unite them into a straight chain or an endless chain, depending upon the cost. What do you figure the cost to be in each case, provided he pays 2 cents for each cut and 2 cents for each weld?

A PRINTER'S MISTAKE

- + 45. In setting up $2^5 \cdot 9^2$ the printer got it 2592. Strange to say this blunder did not affect the value of the result. Can you find another number of four digits which has this peculiarity?

DIOPHANTINE ANALYSIS

- + 46. Find a square proper fraction which if subtracted from unity will leave for a remainder a square proper fraction.

THE BROKEN WEIGHT

47. A drummer while in a grocery store picked up a sledge hammer and broke a rock on the floor into 4 pieces. The merchant was provoked and exclaimed, "I have used that rock in weighing on these balances for many years and it will be difficult for it to be replaced." The drummer then exclaimed, "You have suffered no loss for I can take the four pieces and weigh any amount on these balances from 1 to 40 pounds respectively." Find the weight of each of the four rocks into which the rock was broken.

THE THREE WOMEN AT MARKET

48. Three women, the first with 10 eggs, the second with 30, and the third with 50, went to market. They each got the same price for their eggs, and all returned with the same money. What did each get?

THE STOLEN BAG OF APPLES

49. One night three men, A, B, and C, stole a bag of apples and hid them in a barn over night, intending to meet in the morning to divide them equally. Some time before morning A went to the barn and divided them into three equal shares and had one apple too many, which he threw away. A took

one share and put the others back into the bag. Later B came and did exactly as A had done. Then came C, who repeated what A and B had done before him. In the morning the three met saying nothing of what they had done during the night. The remaining apples were divided into three equal shares with still one apple too many. How many apples were there in the bag at the beginning?

WHAT SHOULD A'S RENT BE?

50. A contracts to raise corn on the shares. He agrees to give $\frac{2}{3}$ if the crop yields 40 bushels per acre and $\frac{1}{2}$ if the yield should be 80 bushels per acre. What in equity should the rent be in bushels per acre if the yield is 60 bushels?

THE DUMB MAN'S ANSWER

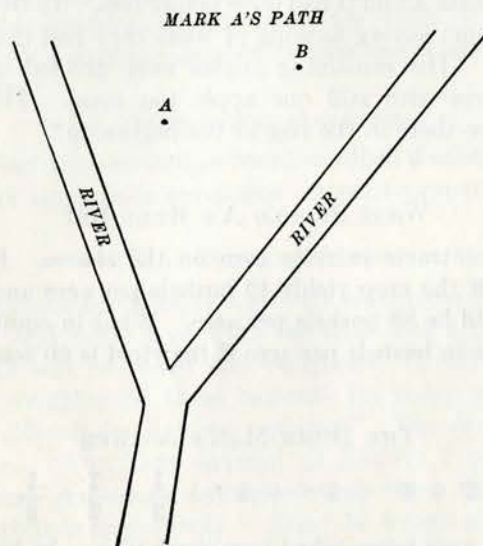
+ 51. $2^2 + 2^{-2} + 2^{2-2} + .2^{-2} + \frac{1}{.2} + \frac{1}{\frac{1}{2}} + \frac{.1}{2}$

A dumb man being asked how much money he had picked up a pencil and wrote the above figures, stating that it expressed his exact worth in dollars. What was his worth?

CAN YOU ASSIST THE PROFESSOR?

- + 52. One day while resting under a tree in my garden, which was surrounded by four straight walls 40, 50, 60, and 70 rods in length, I was asked by a professor, "How much land have you here?" I exclaimed that the tree was equidistant from the four corners and stated that I had never been able to calculate the exact area. I then said, "Now, Professor, I ask you to figure it out for me." He scratched his head and exclaimed, "Impossible! Impossible!" Just how much land did I have is still the question.

53. A and B live between two rivers which meet. A wishes to visit B and desires to take the shortest possible



path so as to carry B a bucket of water from each river. Mark his path.

DIVIDING A DOLLAR

54. Divide a dollar among four men, giving the first one third, the second one fourth, the third one fifth, and the fourth one sixth.

AN AGE PROBLEM

55. A person finds on a certain year that the sum of all the figures in that year is just equal to his age. He also finds that the sum of the figures of the year in which he was born is also equal to this number, and when he is twice this age, it will again be true. How old is this person now?

HOW DO I KNOW YOUR ANSWER?

56. Select a number less than 10. Add 3 to it. Multiply the sum by 2. Divide the product by 2. Subtract from the quotient the number that you selected. Your answer is 3.

THE TURKS AND CHRISTIANS

57. This Recreation is usually proposed in these terms: "It once happened that fifteen Christians and fifteen Turks being in a ship at sea, in a violent tempest, it was deemed necessary to throw *half* the number of persons overboard, in order to disburden the ship, and save the rest. To effect this, with some *appearance* of equity, it was agreed to be done by *lot*, in such a manner that, the persons being placed in a ring, every ninth man should be cast into the sea, until one half of them were thrown overboard. Now the Pilot, being a Christian, was desirous of saving those of his own *persuasion*. How ought he therefore to dispose of the crew, so that the lot might always fall upon the Turks?"

This question may be resolved by placing the men according to the *numbers* annexed to the *vowels* in the words of the following verse:

Po-pu-le-am jir-gam Ma-ter Re-gi-na fe-re-bat,
4 5 2 1 3 1 1 2 2 3 1 2 2 1

from which it appears, that you must place four of those you would save first; then five of those you would have perish. After this, *two* of those to be saved, and then one who is to perish, and so on. When this is done, you must enter the ring formed by the men, and beginning with the first of the *four* men you intend to save, count on to the ninth man, and turn him out to be punished; then counting on, in like manner, to the next *ninth* man, turn him out to be punished in like manner, and so on for the rest, until the fifteen are turned out.

It is reported of Josephus, the author of the Jewish History, that he escaped death by practically working this problem; for being governor of Joppa at the time it was taken by the Roman emperor Vespasian, he was obliged to secrete himself with thirty or forty of his soldiers in a cave, where they made a firm resolution to perish by famine rather than fall into the hands of the conqueror. But being at length driven to great distress, they would have destroyed each other for sustenance, had not Josephus persuaded them to die by *lot*; which he so ordered, that all of them were killed except himself and another, whom he might easily destroy, or persuade to yield to the Romans.

A DITCH PROBLEM

58. A, B, and C contract to dig a ditch 240 rods long. They find the land first consists of rock, then clay for a certain distance, and then sand on to the end of the ditch. If all the land were rock, A could dig it in 90 days, B in 125 days, and C in 34 days. If all were clay, A could dig it in 80 days, B in 120 days, and C in 125 days. If all the land were sand, A could dig it alone in 60 days, B in 48 days, and C in 70 days. C begins at the rocky end, B at the sandy end, and A at the clay soil. When the work is completed, it is found that C dug the rock and that he dug as many rods of rock as A and B combined dug of clay and sand. Determine the number of rods of rock, clay, and sand.

EXPLAIN THE FOLLOWING MAGIC

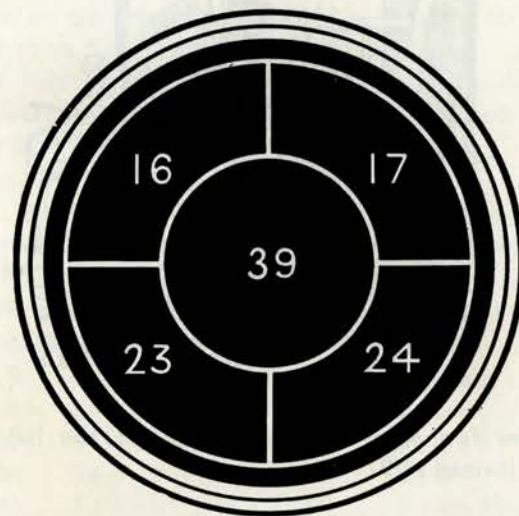
+ 59. Choose any number except 0 or 1, and perform the following operations in succession:

- subtract from 1.
- find the reciprocal.
- subtract from 1.

- find the reciprocal.
- subtract from 1.
- find the reciprocal.
- Multiply by 2 (or any other number the proposer may select). Now, if you have made no mistake in your work, I will give you the number selected promptly on receipt of your result.

TARGET PRACTICE

60. Practicing with my rifle at this target, I scored exactly 100.



How many shots had I fired, and in which parts had they hit the target?

How close can you come to scoring a total of 100 taking as many shots as you please? If it will assist you use 6 numbers on the target instead of 5 by adding 40. If at first you don't succeed try, try again.

THE HOD CARRIER'S PROBLEM

61. (a) How few steps need be taken on this ladder to go up and down, and up to the top, stepping twice on every step?



(b) How few steps will be required if the ladder has 11 rungs instead of 9?

FIND THE NUMBER

62. If a number be divided by 4 more than itself, the quotient is $\frac{3}{4}$. Find the number.

63. A number of three digits in the scale of 7 when expressed in the scale of 9 has its digits reversed. Find the number.

NUTS FOR THE MAGICIAN

THE NUMBER NINE

1. The number nine possesses the most remarkable properties of any of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9. Many of these properties have been known for centuries and have excited much interest among both mathematicians and scholars in general. Because of its wonderful properties the number nine has been regarded as magical and is often referred to as the "magical number."

If we were using the duodecimal scale to-day instead of the decimal, to what number would all the properties of the number nine belong?

PROPERTIES OF NUMBER NINE

2. Some of the properties of the "magic" number 9 are as follows:

(a) *The sum of the digits of any number which is a multiple of 9 is either 9 or a multiple of 9. Thus,*

$1 \times 9 = 9$	$9 = 9$	$7 \times 9 = 63$	$6 + 3 = 9$
$2 \times 9 = 18$	$1 + 8 = 9$	$8 \times 9 = 72$	$7 + 2 = 9$
$3 \times 9 = 27$	$2 + 7 = 9$	$9 \times 9 = 81$	$8 + 1 = 9$
$4 \times 9 = 36$	$3 + 6 = 9$	$10 \times 9 = 90$	$9 + 0 = 9$
$5 \times 9 = 45$	$4 + 5 = 9$	$11 \times 9 = 99$	$9 + 9 = 18 = 1 + 8 = 9$
$6 \times 9 = 54$	$5 + 4 = 9$	$12 \times 9 = 108$	$1 + 0 + 8 = 9$

$$435 \times 9 = 3915.$$

$$3915 = 3 + 9 + 1 + 5 = 18 = 1 + 8 = 9.$$

This is true for every multiple of 9. When we have the factor 9 in a number, it will cling to the expression and will appear at unexpected times and in a variety of ways. This

is because of its relation to the numerical scale. Whatever you do, the figure 9 is sure to turn up again. It is impossible to get rid of it.

(b) *If you take a number consisting of any number of digits and change their order as you please, the number thus obtained for any arrangement of digits, when divided by 9, will leave the same remainder.*

Thus 2345, 3245, 5234, etc. when divided by 9 in each case gives the same remainder, 5. This follows because the sum of the digits is the same for every arrangement. The remainder from dividing a number by 9 is the same as the remainder from dividing the sum of its digits by 9.

(c) *Take a number consisting of any number of digits. Form a new number by reversing the order of the digits. The difference is always divisible by 9.*

(d) *Take the sum of the digits from any number and the difference will be divisible by 9.*

(e) *Take two numbers in which the sums of the digits are the same, the difference of the two numbers will be divisible by 9.*

CASTING OUT NINES

3. Of all the curious properties of the number 9 perhaps the most practical application is the check on Addition, Division, and Multiplication by casting out 9's, also called The Hindu check.

The following illustrations will make clear the method of applying this check.

(a) Addition	Excesses	Add the digits in each of the addends and divide the sum by 9 (or cast out the nines). The remainder is called "the excess." It is the remainder which would be found if the entire number were divided by 9.
4328	8	
5964	6	
3456	0	
5091	6	
2163	3	
<u>21002</u>	<u>23</u>	<u>23 = Sum of Excesses</u>

Excess in the sum $(21002) = 5$.

Excess in the sum of the excesses $(23) = 5$.

The excess in the sum must equal the excess in the sum of the excesses.

(b) *Multiplication*

Excess

$$\begin{array}{r} 3725 \\ 439 \\ \hline 33525 \\ 11175 \\ \hline 14900 \\ \hline 1635275 \end{array}$$

8 = excess of nines in 3725

7 = excess of nines in 439.

$\overline{56}$ = product of excesses

Excess of nines in the product $(1635275) = 2$

Excess of nines in the product of excesses = 2

The excess of 9's in the product must equal the excess in the product of the excesses of the multiplicand and the multiplier.

Results in Division and Subtraction may be checked in a similar way.

GIVEN ONE DIGIT OF REMAINDER I WILL NAME THE OTHER

4. Take any number consisting of two digits, the difference between them being not less than one. Form a new number by reversing the order of the digits. Take the difference between the numbers. Give me one digit of the remainder and I will give you the other.

Thus $83 - 38 = 45$.

GIVEN THE DIFFERENCE BETWEEN THE 1ST AND 3RD DIGITS, THE RESULT IS KNOWN

5. Take any number consisting of three digits, the difference between the first and last being not less than 1. Form a new number by reversing the digits. Find the difference

and divide by 9. The quotient may be read forward, or backward. Thus

$$\begin{aligned} 825 &= \text{Number selected} \\ 528 &= \text{No. Rev. digits} \\ 9 \overline{)297} &= \text{Difference} \\ 33 &= \text{Result} \end{aligned}$$

REVEALING THE ERASED FIGURE

6. Write a number consisting of three or more figures. Divide by 9, and tell me the remainder. Now erase one figure in the given number, not zero. Divide the resulting number by 9. Tell me the remainder and I will tell you the figure erased.

FILLING IN THE FIGURE NECESSARY TO MAKE A NUMBER DIVISIBLE BY 9

7. Write a number with a missing figure and I will immediately fill in the figure necessary to make the number divisible by 9.

Thus, $834632x589$ is a number in which x denotes the missing figure.

FINDING THE CANCELED DIGIT #1

8. Write any integer, the difference between the first and last digits being not less than 1. Form another number by reversing the order of the digits. Subtract and cancel any digit in the result except an 0. Give me the sum of the remaining digits and I will give you the digit canceled.

Ex. Suppose 65423 is the number written by my friend. Then

$$\begin{aligned} 65423 &= \text{Number written} \\ 32456 &= \text{Number (rev. digits)} \\ \hline 32967 &= 21, \text{ sum of remaining digits} \end{aligned}$$

I am not supposed to see the work or know a figure used by him. I only know the sum of the remaining digits, which is 21.

FINDING THE CANCELED DIGIT #2

9. Write any integer, the difference between the first and last digits being not less than 1. Subtract the sum of the digits. Cancel any digit in the result except an 0. Give me the sum of the remaining digits and I will give you the canceled digit.

REVEALING THE REMAINDER

10. Write any integer. Divide it by 9, and give me the remainder. Multiply the original number by the number which I name. Divide the product by 9 and I will reveal the remainder, not seeing any of your work. Thus,

$$\begin{array}{r} 9 \overline{)2468} \\ 274 - 2 = 1\text{st Rem.} \end{array} \qquad \begin{array}{r} 2468 \\ \underline{6} \\ 9 \overline{)14808} \\ 1645 - 3 = 2\text{nd Rem.} \end{array}$$

GIVEN THE LAST DIGIT I WILL REVEAL THE OTHER TWO

11. Take any three-figure number, the first and last figures being unequal. Form another number by reversing the digits. Find the difference between the two numbers. Tell me the last figure and I will tell you the other two.

THE ANSWER REVEALED NOT KNOWING A FIGURE USED

12. Take any number (n) of three digits in which the difference between the first and last digits exceeds 1. Form a new number (n_1) by reversing the order of the digits. Find the difference (d) between the two numbers (n and n_1), form another number (n_2) by reversing the order of the digits of the difference (d). Then add this number (n_2) to the

difference (d) and I will tell you the result, not knowing a figure you used.

RESULT THE SAME FOR ANY NUMBER SELECTED

13. Let each pupil write any number consisting of three digits such that the digits decrease by one from left to right, as 876. Reverse the order of the digits, giving 678 and subtract. Each pupil will then have the same result.

GIVEN THE LAST TWO DIGITS, I WILL REVEAL THE FIRST TWO

14. Take any number consisting of four figures, where the digits differ by 1, as 8765. Reverse it, giving 5678. Find the difference (3087). Give me the last two digits and I will give you the other two.

DIFFERENT NUMBERS, AND SAME OPERATIONS, GIVE THE SAME RESULT

15. Let each pupil write a number consisting of four digits in descending order. Reverse the order of the digits and subtract. Again reverse the order of the digits and add. The result is always the same. Why?

The result in the above is always 10890. The teacher may modify this exercise by having the class take the following additional steps. Subtract 889 from the result. Reverse the order of the digits and subtract. In this case the result is always the same.

If no mistakes have been made, each pupil will have the same result, being expressed by a very familiar character which sometimes is used by teachers when grading or decorating examination papers.

REVEALING A NUMBER SELECTED BY A FRIEND

16. Ask your friend to think of a number. Have him square it. Have him square the next larger number. Then

have him tell you the difference between the squares and you will tell him the number.

REVEALING TWO NUMBERS KNOWN ONLY TO YOUR FRIEND

17. Have your friend write two numbers, each less than 9. Have him double the number on the left, add 1, multiply by 5, add the second number, add 106. Ask him to give you the result, promising in turn to reveal to him the two numbers selected.

REVEALING THREE NUMBERS AND THE ORDER IN WHICH THEY ARE WRITTEN

18. Write any three numbers of one digit each. Double the one on the left hand; add 5; multiply by 5; add the middle number; multiply by 10; add the third number. Give me the result and I will reveal the three numbers to you in their order.

REVEALING FOUR UNKNOWN NUMBERS

19. (a) Select any four numbers of one digit each. (b) Double the one on the left-hand side and add 1. (c) Multiply the result by 5 and add the second. (d) Double the result and add 1. (e) Multiply the result by 5 and add the third. (f) Double the result and add 1. (g) Multiply by 5 and add the fourth. (h) Give me the result and I will reveal to you the four numbers in their order.

FINDING THE DIFFERENCE IN AGE BETWEEN YOURSELF AND AN OLDER PERSON

20. The difference in age between yourself and another person who is older than you may be easily found as follows:

(a) Write down a number containing as many nines as there are figures in your age.

(b) Subtract your age.

- (c) Tell the other person to add the result to his age.
 (d) Tell him to cancel the left-hand figure and add it to the right-hand figure.
 (e) The result is the difference in age.

Ex. Suppose your age is 38 and the age of the older person is 43. Find the difference, the latter age being unknown to you.

Solution:

- | | |
|--|-----|
| (a) Write down | 99 |
| (b) Subtract your age | 38 |
| (c) Have your older friend add his age | 61 |
| to the result | 43 |
| (d) Cancel 1 and add it to 4 | 104 |
| | 1 |
| (e) The difference in ages then is | 5 |

THE RESULT REVEALED, KNOWING ONLY THE NUMBER OF DIGITS

21. Write a number consisting of not more than ten nor less than two digits, differing by one and arranged in descending order, such as 5432. Form a new number (2345) by reversing the order of the digits. Find the difference between these numbers. I will then reveal this difference to you if you will only give me the number of digits in the original number selected.

REVEALING A CANCELED DIGIT IN A SUM

22. You write two or more numbers containing the same number of digits. I will then write the same number of numbers. I will then retire, asking you to cancel any figure except an 0, and then find the sum of the remaining figures. If you will then give me the sum of the figures in the sum, I will give you the figure you canceled.

For example, suppose the following numbers are written.

$$\begin{array}{r} 842 \\ 253 \\ \hline 746 \\ 157 \\ \hline 1298 \end{array} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{The numbers written by you.} \\ \text{The numbers written by me.} \\ \text{Sum of remaining figures.} \end{array}$$

$1 + 2 + 9 + 8 = 20$, the sum of the figures in the sum, which you give me.

THE PROFESSOR'S TRICK

23. A professor to amuse his class of four pupils showed them the number 39,996 on a card which he said was the sum of eight numbers each consisting of four digits. He asked each pupil to write a number, stating that he would then write the other four.

$$\begin{array}{r} 5364 \\ 2108 \\ 6314 \\ 4537 \\ \hline \end{array} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Numbers} \\ \text{written by} \\ \text{the pupils.} \end{array}$$

$$\begin{array}{r} 5462 \\ 3685 \\ 7891 \\ 4635 \\ \hline \end{array} \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \begin{array}{l} \text{Numbers} \\ \text{written by} \\ \text{the professor.} \end{array}$$

39,996 The number proposed

The upper numbers were written by the pupils. The professor then wrote the four numbers as shown and assured them that the total would be the number shown them on the card. Show how he did this.

A MYSTERIOUS GROUP OF NUMBERS

24. The following numbers are arranged in 6 horizontal rows, 6 vertical columns, and 2 diagonals, forming 14 numbers

of 6 digits each. By reading these numbers backward or forward we have 28 numbers of 6 digits each.

2	3	4	8	6	4
4	7	2	3	6	5
8	2	4	1	4	8
6	5	8	4	1	3
5	6	2	3	7	4
2	4	7	8	3	3

You may choose any one of the following and I will restore the figure or figures erased.

- (1) Erase any figure.
- (2) Erase any two figures.
- (3) Erase any three figures.
- (4) Erase any four figures.
- (5) Erase any five figures.
- (6) Erase any figure in each column.
- (7) Erase any figure in each row.
- (8) Erase any figure in each diagonal.
- (9) Erase all figures in any column.
- (10) Erase all figures in any row.
- (11) Erase all figures in either diagonal.
- (12) Erase all figures in any column and diagonal.
- (13) Erase all figures in any row and diagonal.
- (14) Erase all figures in both diagonals.
- (15) Rearrange the digits of any row, column, or diagonal, omitting one digit.

THE MYSTERIOUS GROUP OF NUMBERS — (Continued)

25. You may choose any one of the following. In such case cancel any figure in the result except an 0. Give me the other figures and I will reveal the canceled figure.

- (1) Select any row, column, or diagonal and multiply it by any integer.

- (2) Select any two rows or columns and take their difference.
- (3) Take the sum of any number of rows or columns.
- (4) From the sum of any number of rows, columns, or diagonals subtract any row, column, or diagonal.
- (5) From the sum of the six rows subtract the sum of the digits of two or more rows, columns, or diagonals.
- (6) Change each of the six numbers of six digits given to numbers of only five digits by erasing any column and find their sum.

NOTE. Enough has been said about the group of mysterious figures, but possibly you would like to arrange a similar group of mysterious figures to be used for the entertainment of your friends.

FUN WITH FIGURES

26. A little girl said to her teacher, "Professor, permit me to be teacher just for a few moments." "Good," said the professor. "Acting student reminds me of those 'good old school days, golden rule days.' What will you have me do?" he exclaimed.

(1) "I will ask you to write two or more numbers on the board. For convenience use numbers consisting of the same number of digits. When you have finished writing them, I will write a number. You may next have the class blindfold me. I will ask you then to cancel a figure — any one of the nine digits. Add the other figures and give me the sum of the digits in the sum. I shall then promptly name the figure you canceled." He did as requested and to the astonishment of all she named the figure canceled.

The figures placed on the board are given below.

3146	Numbers written by the professor
4725	
6212	
5825	The number written by the little girl
19,208	= sum

$$1 + 9 + 2 + 0 + 8 = 20, \text{ sum of the digits}$$

(2) She then said, "Now, professor, write two or more columns of figures and I will write a single column either to the left or to the right of yours — just as you say." At his suggestion she placed it on the left.

2	4	2	1	The three columns on the right
6	3	5	4	were written by the professor while
8	2	1	7	the one on the left was written by
2	6	0	1	the little girl.
8	3	2	5	
0	8	1	9	
28	6	8	7	= sum

$$2 + 8 + 6 + 8 + 7 = 31, \text{ sum of the digits}$$

He again canceled a figure, added, and gave her the sum of the digits. They were greatly amazed when again she made known the canceled figure.

(3) "I wish to make another suggestion," she said. "Write any number of numbers, or any number of columns of figures. I will then write a single number, or column consisting of only three figures and will place it above, below, on the left, or on the right of your numbers — just wherever you say." The figures used this time were

1	8	324
6	1	245
5	5	3865
6	6	4521
2	9	3706
1	6	1661 = sum

$$1 + 6 + 1 + 6 + 6 + 1 = 21, \text{ sum of the digits}$$

She first wrote the column 16562 but being reminded that she had promised to use only three figures, added the 1, 6, and 2 and wrote the numbers 5, 6, and 9, marking out the

column 16562. She then stated, "It was not necessary to write three figures. I could have written only two and just as easily named the canceled figure." By this time both the instructor and class were completely amazed and puzzled at the statements made by the little girl.

(4) Before taking her seat she remarked to the instructor: "It was unnecessary for me to write more than one figure. I can write a single figure and will place it anywhere you say on the board. I can then find the canceled figure just as easily as by writing a column consisting of many figures." "Seeing is believing," exclaimed the professor. So she placed the single figure on the board. He added the number in with the others and gave her the sum of the digits. To the utter amazement of every one the canceled figure was again called out.

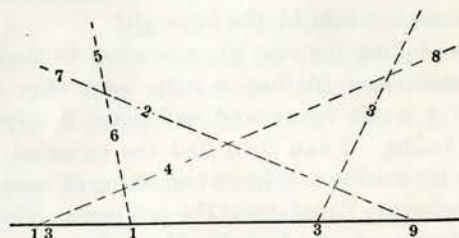
In conclusion, you are asked to explain how the little girl found the figures she wrote in (1), (2), (3), and (4). Also tell how she revealed the canceled figure in each case.

THE BOY AND HIS BLOCKS

27. Santa brought a little boy a number of boxes of blocks. Each contained eight blocks bearing the numbers 1, 2, 3, 4, 5, 6, 7, and 8 respectively. One day he found a certain block missing. He was worried and went to his father with his trouble. As father questioned him, he said, "Father, one of my blocks is missing, but I do not know its number." The others are here except those bearing the numbers 4 and 5. They are in the other room. The father replied, "Son, we will not need the other blocks to find the number of the missing one. Stack the blocks you have in two or more columns, add as you did at school, give me the sum, and I will tell you the number on the missing block." How did the father find the number on the lost block?

MYSTERIOUS ADDITION

28. Place any eight of the nine digits anywhere in a plane, as indicated below. Add these in columns in any order, as



indicated by dotted lines. Give me the sum and I will then name the digit not used.

THE NINE DIGITS

29. (a) Arrange the nine digits 1, 2, 3, 4, 5, 6, 7, 8, 9 in two or more columns and add. To this sum add the sum of 9 and any digit. Give me the sum and I will name the digit you added to 9.

(b) Arrange the nine digits in two or more columns and add. From the sum subtract the sum of 9 and any digit. Give me the remainder and I will name the digit you added to 9.

A SCHOOL MARM TO HER CLASS #1

30. "How many digits are there, pupils?" "Ten," was the reply. "Name them," she then asked. "1, 2, 3, 4, 5, 6, 7, 8, 9, 0," was the response. She then remarked, "The 0 has no value; we will throw it away. The 9 is called the magic number. I am afraid of it, so we will also throw it away."

"Now of the eight digits remaining I will ask each of you to pick out your favorite one and place it in your tablet so

that no one can see it or know what it is. Then each of you take the seven remaining digits and arrange them in two or more columns. Find their sum. Then find the sum of the digits.

"When each of you have had sufficient time to find the sum of the digits, I will ask you to write it on a piece of paper bearing your name. I will then read each name to the class and name each one's favorite digit."

THE SCHOOL MARM TO HER CLASS #2

31. The school marm said, "Now, pupils, we will try something different. A moment ago we discarded the digits 0 and 9. We will again discard them, but in addition to this we will let the boys also discard the digits 2 and 7 while the girls will discard the digits 4 and 5.

"I will ask each of you to set apart, as before, one of the six remaining digits. Handle the other five digits exactly as you did before. I will then collect the papers and reveal to each one the digit you set apart."

HOW TO TELL A PERSON'S AGE AND MONTH OF BIRTH

32. Have the person whose age is to be discovered do the figuring.

For example, if it is a lady, suppose her age is 31, and that she was born in December. Let her put down the number of the month in which she was born and proceed as follows:

Number of month of birth	12
Multiply by 2	24
Add 5	29
Multiply by 50	1450
Then add her age, 31	1481
Subtract 365, leaving	1116
Now add 115	1231

She then announces the result, 1231, whereupon she may be informed that her age is 31, and December, or the twelfth month, is the month of her birth. The two figures on the right in the result will always indicate the age, and the remaining figure or figures the month in which her birthday comes.

A QUEER TRICK OF FIGURES

33. A man with an uncanny mania for juggling with figures placed a pad of paper and a pencil in his friend's hand, and said:

"Put down the number of your living brothers.

Double the number.

Add 3.

Multiply the result by 5.

Add the number of your living sisters.

Multiply the result by 10.

Add the number of dead brothers and sisters.

Subtract 150 from the result."

The friend did as directed.

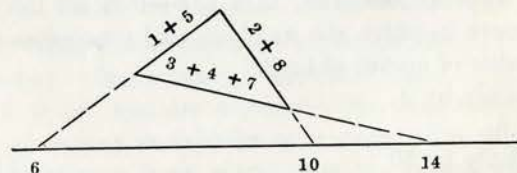
"Now," said the other with a cunning smile,

"The right-hand figure will be the number of deaths.

The middle figure will be the number of living sisters.

The left-hand figure will be the number of living brothers."

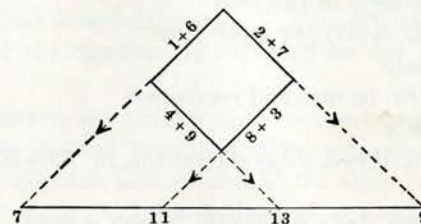
A QUEER WAY OF FINDING A MISSING DIGIT



34. Have a friend write any seven of the first eight digits on the sides of a triangle, find the sum of those on each side, and combine the results, forming a number such as 61014,

which we will call the sum. Name by inspection the digit omitted by your friend when given only a number such as 61014.

ANOTHER WAY OF FINDING A MISSING DIGIT



35. Have a friend write any eight of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 on the sides of a quadrilateral or polygon. Have him add the digits on each side. Then have him combine the results as above, giving a number such as 711139, which we will call the sum. You are to name the digit omitted by your friend, knowing only a number such as 711139.

YOUR FAVORITE NUMBER

36. Many persons have what they consider a lucky or favorite number. Show such a person the row of figures,

1, 2, 3, 4, 5, 6, 7, 9

consisting of the numbers from 1 to 9 inclusive with the 8 only omitted. Then ask your friend to name his favorite number. He may select any number he pleases from 1 to 9, say 3. You then reply that as he is fond of threes, he shall have plenty of them and accordingly proceed to multiply the series given above by such a number that the product will consist of threes only.

Required, to find the multiplier for each number that may be selected, to produce a result containing only the person's favorite number.

A BOY AND HIS EMPLOYER

37. A boy asked for an increase in wages, whereupon his employer proceeded to show him that he was not really entitled to what he was getting. He reasoned as follows:

There are 365 days in the year	365 days
You sleep 8 hr. a day, or $\frac{1}{3}$ the time	<u>122</u>
This leaves only	243
You spend 8 hr. in rest and recreation	<u>122</u>
This leaves only	121
You are off Sundays & $\frac{1}{2}$ day on Saturdays	52+26, or <u>78</u>
This leaves	43
You have $1\frac{1}{2}$ hr. daily for lunch, 5 days a week	<u>16</u>
This leaves	27
You are given 3 weeks vacation	<u>21</u>
This leaves	6
There are 6 holidays — New Year's Day, Washington's Birthday, July 4, Armistice Day, Thanksgiving Day, and Christmas Day	<u>6</u>
So you do not work at all	0
Moreover, there are Columbus Day, Memorial Day, and Labor Day; so you owe me for those.	
Can you find the fallacy for the young man?	

A MYSTIFYING STUNT

38. At a party the other evening the hostess presented a pack of cards, about the size of postals, and allowed each gentleman to take one at random. On each card were typed the following questions:

<i>In what year were you born?</i>	
<i>What is your age?</i>	
<i>In what year did you begin working at your present occupation?</i>	
<i>How many years have you worked at it?</i>	
Total	

"Now," she said, "write in the blanks opposite each question the correct figures. In determining your age, you are to take the difference between the year of your birth and the present year; and similarly in determining the answer to the last question. Then total the figures."

When all had done so, she said: "Turn the cards over, and you will find the correct total written on the back of each one."

To everybody's astonishment it was just as she had said. And their amazement increased when, on comparing cards, they found that each had arrived at the same total.

Try this curious mathematical stunt on yourself and your friends, and see if you can figure out how it works that way.

MATCH PROBLEMS

39. (a) With 6 matches form four equilateral triangles, the side of each being equal to the length of a match.

(b) Form four squares with 9 matches.

OTHER MATCH PROBLEMS

40. Given the following incorrect statements in the form of equations, formed by using matches.

(a) $V I - I V = I X$

(b) $M V - I = I V$

(c) $V - I V = V I I$

In each case you are required to change the statement to a true equation by moving a single match. You are not permitted to move any match which touches another.

THREE PROBLEMS THAT WILL KEEP 'EM GUESSING

41. (a) To one half of a strike
Add two thirds of a ton,
Then a stone you will see,
If 'tis properly done.
- (b) Write fifty; add zero; add five; add one fifth of eight.
The sum has been called "the greatest thing in the world."
- (c) Add one third of 12 to four fifths of seven so as to get 11.

FINDING YOUR COGNOMEN

42. Hand your friend a piece of paper and ask him to write the numbers you request and the result will be the name by which he is popularly known.

- (1) Ask him to write the year in which he was born.
- (2) Add 4.
- (3) Add his age at his birthday in the current year.
- (4) Subtract $\left\{ \begin{array}{l} 27 \text{ during year } 1932. \\ 28 \text{ during year } 1933. \\ 29 \text{ during year } 1934. \\ 30 \text{ during year } 1935. \end{array} \right.$

NOTE. The number to be subtracted during any given year may be obtained as follows: Thus for 1936 use $25 + 6$, or 31. For year 1945 use $25 + 20$, or 45. The number to be added to 25 in each case is found by subtracting 1930 from the current year.

- (5) Multiply the result by 1000.
- (6) Subtract 694,423.
- (7) Have him substitute letters for figures; for 1, the letter *a*; for 2, the letter *b*; for 3, the letter *c*; etc.
- (8) The result will be the name by which he is popularly known — "A Bad Egg."

AN APRIL 1ST PUZZLE

43. Spring this on your class or friends on the 1st day of April:

- (1) Ask each one to write a number of their choice consisting of three digits, the difference between the first and last digit exceeding 1, say 531.
- (2) Reverse the digits, giving 135.
- (3) Find the difference between the numbers, 396.
- (4) Again reverse the digits, giving 693.
- (5) Add, giving 1089.
- (6) Multiply by a million, giving 1,089,000,000.
- (7) Subtract 966,685,433, leaving 122,314,567.
- (8) Substitute the following letters for figures. Under each figure 1, write the letter *l*.
Under each figure 2, write the letter *o*.
Under each figure 3, write the letter *f*.
Under each figure 4, write the letter *i*.
Under each figure 5, write the letter *r*.
Under each figure 6, write the letter *p*.
Under each figure 7, write the letter *a*.
- (9) Have them read the result backward —

THE THREE-GIRL STUNT

44. Three girls were dressed in gymnasium suits wearing the numbers "three," "six" and "one" respectively. The following questions were propounded by them.

- (a) Can you arrange us so that we are divisible by 4?
- (b) Can we be divided by 8? (c) Are we divisible by 23?
- (d) Can you arrange us so as to express 63?
- (e) Can you arrange us so as to express $\frac{1}{2}$? 2?
- (f) Arrange us so as to express the largest number possible.

YOUR YEAR OF BIRTH AND AGE

45. Write the year of your birth.
 Double it.
 Add 5.
 Multiply by 50.
 Add your age.
 Add 365.
 Subtract 615.

If no error has been made your year of birth is on the left and your age on the right.

NOTE. This mystery may be explained as follows:
 Let X = year of birth and Y = age of person. The result obtained by taking the given steps is $50(2X + 5) + Y + 365 - 615$. The number, however, giving the year of birth on the left and age on the right, is $X(100) + Y$.

$\therefore$ the required number is
 $X(100) + Y$, or $50(2X + 5) + Y + 365 - 615$.

TO REVEAL A PERSON'S YEAR OF BIRTH AND AGE

46. Have your friend:

- (a) Write his year of birth.
- (b) Double it.
- (c) Add 5.
- (d) Multiply by 50.
- (e) Add his age.
- (f) Add 861.

Have the person give the result to you. You are then to reveal his year of birth and age.

THE MAGIC AGE TABLE

47. By using this table a person's age can easily be found.
 For a copy of this table see "Mathematical Wrinkles," page 75.

NUTS FOR THE PROFESSOR

ARITHMETIC

a 1. A man on the platform of a station observed that a train passed him in 10 seconds and passed completely through a station, which is 308 yards long, in 24 seconds. How long was the train and how fast was it going?

a 2. Show that, if we reverse the two digits of a number, the difference between the two numbers is always divisible by 9.

a 3. Two trains pass each other in 10 seconds when moving in opposite directions; when moving in the same direction the latter train passes the other in 25 seconds. What is the speed of each, if their respective lengths are 300 feet and 500 feet?

a 4. Divide the twelfth power of 89,798 by the eleventh power of the same number and multiply the quotient by the number whose square root is equal to its cube root.

a 5. How long must a tape be to wind spirally around a cylinder that is 40 feet long and 3 feet in circumference, provided it is to pass once around the cylinder every 4 feet?

a 6. Two automobiles 20 miles apart are approaching each other, each traveling 10 miles per hour. A bee which flies at the rate of 15 miles per hour starts at the radiator of one automobile and flies back and forth between their radiators until the autos meet. How far does the bee fly?

a 7. Suppose 4 apples are worth 5 plums; 3 pears are worth 7 apples; 8 apricots are worth 15 pears. Then if 5 apples sell for 2 cents, what is the smallest sum of money that will buy an equal number of the four kinds of fruit?

a 8. A gives B a mortgage on a farm for \$2500. B gives C a mortgage on the same farm for \$5000. C gives A a mortgage on the same farm for \$10,000. The farm is sold for \$15,000. If B's mortgage is secured before A's and C's before B's, who will lose and how much?

a 9. I marked goods to gain 60%, but on account of using an incorrect measure I gained only 40%. What was the length of the measure used?

a 10. If a rectangular lot is worth \$800, what is a similar lot worth whose length and width are each 50% greater?

a 11. If 6 be $\frac{1}{3}$ of 12, what would $\frac{1}{4}$ of 20 be?

a 12. Mary is 24 years old and is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

a 13. A and B divide a sum of money equally. If A got \$2000 and 10% of the remainder, find the sum.

a 14. Four boys, A, B, C, and D, found \$14. They agreed that D should have 6 times as much as A, because he saw the money first, and B half as much as D, because he picked it up, and C $\frac{1}{3}$ as much as B and D, if he would keep silent. How much did he get for his silence?

a 15. I sold a horse and a cow for \$210. I gained 25% on the horse and lost 25% on the cow. If the cow cost $\frac{2}{3}$ as much as the horse, what was my gain?

a 16. What per cent do I gain if I sell $\frac{5}{8}$ of a bill of goods for what $\frac{3}{4}$ of it cost me?

a 17. A, B, and C, whose rates of walking are $3\frac{1}{2}$, 4, and 5 miles an hour respectively, walk on circular tracks, the circumferences of which are $\frac{1}{5}$, $\frac{1}{4}$, and $\frac{3}{8}$ of a mile respectively,

and the centers of which are on the same straight line. At the same instant they start from points on this line. Find (a) when they will all be on the line at the same time, and (b) when they will all be at the same time at the points from which they started.

a 18. Find the smallest number which is divisible by 13, and which when divided by any of the numbers from 2 to 12 inclusive leaves a remainder of 1.

a 19. A pays \$1.40 for his share of a grindstone 32 inches in diameter and grinds first; then B, who pays \$1.00; and lastly C who pays 80¢. Find the radius of C's stone, if each grinds in proportion to what he pays.

a 20. Three women buy together a ball of silk 6 inches in diameter. How much of the diameter must each wind off to get her third?

a 21. Supposing a given number of marriages contracted between males aged 30 and females aged 25, what per cent of the original number will survive as married couples, widowers, and widows, at the end of 10 years, assuming the probability of dying within 10 years at the age of 30 to be $\frac{2.65}{25.01}$, and at the age of 25 to be $\frac{2.37}{26.11}$?

ALGEBRA

b 1. Factor

(a) $1 + 4x^4$.

(b) $x^{3e} - y^{-3e}$.

(c) $e^{3x} - 2e^x - 24e^{-x}$.

(d) $e^{3x} + e^{2x} + e^x + 1$.

b 2. It is desired to have a 10-gallon mixture of 45% alcohol. Two mixtures, one of 95% alcohol and another of 15% alcohol, are to be used. How many gallons of each will be required to make the desired mixture?

- + b 3. Solve the following equation for n :

$$2^{6n+3} \cdot 4^{3n+6} = (8^n)^n.$$

b 4. Simplify $[(e^x - e^{-x})^2 + 4]^{\frac{1}{2}}$.

b 5. Simplify $\sqrt[3]{192} - 4\sqrt[3]{24} + \sqrt[3]{375}$.

b 6. Simplify $3\sqrt{\frac{7}{2}} + 3\sqrt{\frac{7}{2}} - 2\sqrt{\frac{1}{14}}$.

b 7. Simplify $\frac{3}{\sqrt{5}} - \sqrt{2} + \sqrt{7 + 2\sqrt{10}}$.

b 8. Simplify $\frac{(4 + \sqrt{15})^{\frac{3}{2}} + (4 - \sqrt{15})^{\frac{3}{2}}}{(6 + \sqrt{35})^{\frac{3}{2}} - (6 - \sqrt{35})^{\frac{3}{2}}}$.

b 9. A and B are at different points on a straight road. A travels toward B, and reaches B's original position 11 minutes after B had left. B travels toward A, and reaches A's original position 15 minutes after A had left. Each then starts back, and meet halfway at 4 P.M. When did each start?

b 10. Two men were walking along a railway track, each at the rate of 3 miles per hour, and in opposite directions. A passing train ran completely by one in 5 seconds and by the other in 6 seconds. How many feet long was the train?

b 11. A man was born in the nineteenth century. He was x years of age in the year x^2 . Find his age in the year 1875.

b 12. A spherical iron shell 1 ft. in diameter weighs $\frac{9}{216}$ of what it would weigh if solid. Find the thickness of the metal, it being known that the volume of a sphere varies as the cube of its diameter.

b 13. A is three times as old as B was when A was as old as B is now; and in 24 years' time B will be twice as old as A was when B was half as old as A is now. Find the ages of A and B.

b 14. A body is composed partly of copper and partly of tin. If the copper had been tin, and the tin copper, the

weight of the body would have been greater by 9% than what it actually is. The weights of equal volumes of copper and tin are as 8.96 to 7.29. How much of the body is copper and how much tin?

+ b 15. Solve the equation, $2^x + 2^{x-1} = 10$.

b 16. A blacksmith had a stone weighing 40 pounds and a skilled mason broke it into 4 pieces whereby any number of pounds from 1 to 40 could be weighed on scales. Find the weight of each of the four pieces.

b 17. A man bought a horse for a dollars and agreed to pay interest and principal in n equal monthly installments, interest r per cent per annum. Show how to compute r and n .

As a practical example, a loan company will lend \$1200 to be repaid in 120 monthly installments of \$12.50 each. What is the rate of interest?

b 18. Find three rational right triangles whose sides are integers and areas equal.

+ b 19. Solve the equation $2^{2x} - 21 \cdot 2^x + 80 = 0$.

b 20. Hanging over a pulley there is a rope with a weight at one end; at the other end hangs a monkey of equal weight. The rope weighs 4 oz. per foot. The combined ages of the monkey and its mother are 4 years and the weight of the monkey is as many pounds as its mother is years old. The mother is twice as old as the monkey was when the mother was half as old as the monkey will be when the monkey is 3 times as old as its mother was when she was 3 times as old as the monkey was. The weight of the rope and weight is half as much again as the difference between the weight of the weight and the weight of the weight plus the weight of the monkey. What is the length of the rope?

b 21. Conditions are such that a ball falls 50 feet and rebounds 25 feet; then falls 25 feet and rebounds 12.5 feet, and so on. Assume that the ball comes to rest. Find the distance through which it moves, and the time it takes to come to rest. Use $g = 32.2$.

+ *b* 22. Find all solutions of the equation $x^{\sqrt{x}} = x^x$.

PLANE GEOMETRY

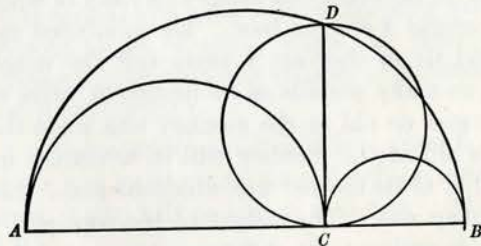
c 1. Three circles are tangent externally at the points A , B , and C , and the chords AB and AC are produced to cut the circle BC at D and E . Prove that DE is a diameter.

c 2. Each angle formed by joining the feet of the three altitudes of a triangle is bisected by the corresponding altitude.

c 3. Construct a triangle, having given a , b , m_c .

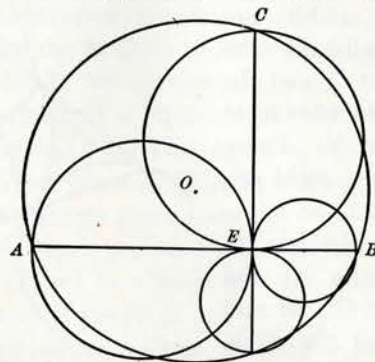
c 4. Construct a triangle, having given m_a , m_b , m_c .

c 5. If C is a point in the straight line AB , the three semicircles, drawn respectively upon AB , AC , and CB as diameters,



bound an area equivalent to a circle whose diameter is the perpendicular CD , D being in the largest semicircle.

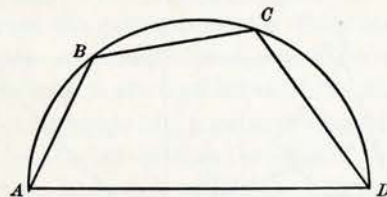
c 6. If two chords of a circle are perpendicular to each other, the sum of the four circles described on the four segments as diameters is equivalent to the given circle.



c 7. Construct the maximum equilateral triangle through three given points.

c 8. If a , b , and c denote the sides of a triangle, and R the radius of the circumscribing circle, show that the area of the triangle is $\frac{abc}{4R}$.

c 9. Three chords whose lengths are 6, 8, and 10, just go around in a semicircle. Find the radius of the circle.

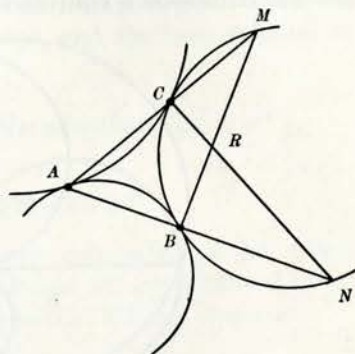


c 10. To construct a triangle having given the three feet of the altitudes.

c 11. The area of the ring included between two concentric circles is equal to that of a circle whose radius is one half a chord of the outer circle drawn tangent to the inner.

c 12. Given two concentric circles. To draw a chord in the larger circle so that it equals twice the chord formed in the smaller circle.

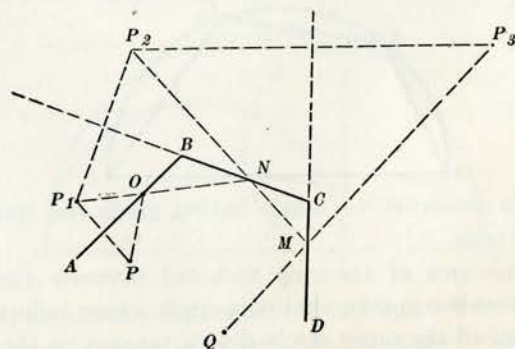
c 13. Three circles are tangent externally at the points A , B , and C , and the chord AB is produced to meet the circle BC at N . Prove that $\triangle ACN$ is a right triangle.



c 14. Three circles are tangent externally at the points A , B , and C . M and N are points on the circle CB , being determined by producing AB and AC . Draw the lines MB and CN intersecting in R . Prove that the quadrilateral $ABRC$ is concyclic.

c 15. To find the point in a given line AB , the sum of whose distances from two given points P and Q is a minimum.

c 16. To find the shortest distance from P to Q en route of the lines AB , BC , and CD , i.e., the path of the ray reflected by the successive surfaces AB , BC , and CD .



c 17. Divide a circle into

- (i) two equal parts
- (ii) three equivalent parts
- (iii) four equivalent parts

In each case the division must be made so that each part will be bounded by semicircles only.

c 18. To construct a circle which will pass through two given points A and B and be tangent to a given line CD .

c 19. To construct a circle passing through a given point P and tangent to two given lines AB and CD .

c 20. To construct a circle which will pass through a given point P , be tangent to a given line AB , and also be tangent to a given circle of center O .

c 21. To construct a circle which will pass through two given points A and B and be tangent to a given circle O .

c 22. To construct a circle passing through a given point P and tangent to two given circles O and O' .

c 23. Construct a triangle ABC , having given the distance of its incenter, circumcenter, and orthocenter from the vertex A .

c 24. The median to one side of a triangle is less than one-half the sum of the other two sides.

c 25. If from the extremities of a diameter perpendiculars be drawn upon any chord (produced if necessary), the feet of the perpendiculars are equidistant from the center.

c 26. Given an angle and a point without its sides to draw through the point a line cutting the sides of the angle, forming a triangle whose perimeter equals the length of a given line.

SOLID GEOMETRY

d 1. Given seven lines, no three of which lie in the same plane. If they pass through the same point, how many planes are determined by them?

d 2. A solid cubical block of metal weighed a ton, but after a square hole was cut completely through the metal, parallel to a face, the weight was only 1280 pounds. Find the size of the hole, if a cubic inch of the metal weighs a quarter of a pound.

d 3. A square and its circumscribing circle revolve about the diagonal of the square as an axis. Compare the volumes and surfaces of the solids generated, the diagonal being 6 feet.

d 4. Find the area of the zone of a sphere of radius R , illuminated by a lamp placed at the distance h from the sphere.

d 5. How much of the earth's surface would a man see if he were raised to the height of the radius above it?

+ *d 6.* To what height must a man be raised above the earth in order that he may see one sixth of its surface?

d 7. A 6-inch hole is cut through the center of a 10-inch sphere. What part of the sphere remains?

d 8. Given $O-ABC$, a tetrahedron with tri-rectangular trihedral angle at O . Prove that the square of the area of face ABC equals the sum of the squares of the areas of faces AOB , BOC , and AOC .

ANALYTICAL GEOMETRY

e 1. If (2, 6), (7, 8), and (-7, 11) are the three vertices of a parallelogram, find the three other possible vertices.

e 2. Find the equation of the line through the points (2, 1) and (3, -2).

+ *e 3.* The equation of a chord is $\frac{x}{a} + \frac{y}{b} = 1$ and the equation of the circle is $x^2 + y^2 = r^2$. Find the length of the chord.

e 4. Find the distance from the origin to the line
 $ax + by = c^2$.

e 5. Find the equation of a circle passing through the origin, and cutting the lengths a and b from the axes.

e 6. Find the equation of a circle having for a diameter the common chord of the circles

$$x^2 + y^2 = r^2 \text{ and } (x - a)^2 + y^2 = r^2.$$

e 7. The perpendicular bisectors of the sides of a triangle meet in a point.

e 8. The lines joining the vertices of a triangle to the points of contact of the inscribed circle are concurrent.

e 9. If a, b, c are the sides of a triangle, and $5(a^2 + b^2 + c^2) = 6(ab + bc + ca)$, show that the incircle passes through the centroid of the triangle.

e 10. A and B are two fixed points; CD is a line parallel to AB . x and y are two variable points on CD such that xy is of constant length. Ax and By meet in P . Show that the locus of P is a straight line parallel to AB .

PHYSICS

f 1. If a man up in a balloon
Should fire a level gun at the moon,
How high is he if the bullet and sound
At the same time should reach the ground?

f 2. A rifle was fired in a horizontal direction with a velocity of 900 feet per second, from the top of a vertical cliff 100.5 feet high, standing on the seashore. How far from the base of the cliff did the ball strike?

f 3. Is the Centigrade and Fahrenheit reading ever the same? If so, for what temperature?

f 4. If a man can jump 3 feet high on the earth, how high could he jump on the moon?

f 5. On the top of a tower 200 feet high stands a pole 27 feet high. At the moment a bullet let fall from the top of the pole passes the bottom, a stone is let fall from a point 45 feet below the top of the tower. Where will the bullet overtake the stone?

f 6. A hammer weighing 2 pounds has a velocity of 12 feet per second at the instant it strikes the head of a nail. Find the force which the hammer exerts on the nail if it is driven into the wood $\frac{1}{4}$ of an inch.

f 7. Standing at a certain distance from a well, I observed a man drop a stone into the well. The man dropping the stone heard it strike the bottom one second sooner than I did, and the sound of the stone striking the bottom reached my ear in one half the time that it took the stone to fall to the bottom. Find the depth of the well.

f 8. How high above the earth's surface (radius 4000 mi.) would a pound weight weigh but one ounce avoirdupois by a scale indicator, corrected for change of elasticity by temperature?

f 9. What temperature will result from mixing 100 lb. of ice at 14° with 80 lb. of steam at 270° F.?

f 10. If ice floats with $\frac{1}{10}$ of its volume above the surface of the water, but in alcohol neither rises nor sinks no matter where placed below the surface of the liquid, what is the specific gravity of alcohol?

f 11. Air at 27° centigrade is reduced to half its original volume. What will its temperature be when reduced?

f 12. A supposedly weightless rope passing over a frictionless pulley has a ten-pound weight hanging on one end and a ten-pound monkey on the other. What will happen when the monkey climbs the rope?

f 13. A ball is shot from a gun with a horizontal velocity of 1000 feet, at such an angle that the highest point in its flight is 257.28 feet. What is its range?

f 14. A ball falls from a certain height; 3 seconds after it has started another body falls from the height of 787.92 feet. From what height must the first fall if both are to reach the ground at the same instant?

f 15. A sportsman fires his gun and $2\frac{1}{2}$ seconds later hears its report the second time. The temperature being 0° C., how far away is the reflecting surface?

f 16. How many pounds of steam at 100° will just melt 100 pounds of ice at 0° ?

f 17. A uniform bar of metal 10 inches long weighs 4 pounds. A weight of 6 pounds is hung from one end of the bar. Determine the position of the fulcrum upon which the loaded bar will balance.

f 18. Suppose that when an orchestra has nearly finished a performance, an observer should move away from the orchestra with a velocity twice that of sound. Describe his relation to the sounds previously executed by the orchestra.

f 19. A hollow ball of iron weighs 1 Kg. What must be its least volume to float on water?

f 20. A piece of wood weighing 300 grains has tied to it a piece of lead weighing 600 grains; together they weigh in water 472.5 grains. The density of lead being 11.35: (a) What does the lead weigh in water? (b) What is the density of the wood?

f 21. All other things being equal, which will cause the greater racket, two crying babies at a distance of four feet, or three crying babies at a distance of six feet?

f 22. I dropped a stone in a well and 5 seconds later heard it strike the bottom. How deep is the well?

TRIGONOMETRY

g 1. The sides of a triangle are three consecutive integers, and the largest angle is twice the smallest angle. Find the sides.

+ *g* 2. Find $\sin 3x$ in terms of $\sin x$.

+ *g* 3. Find $\cos 3x$ in terms of $\cos x$.

+ *g* 4. Find $\tan 3x$ in terms of $\tan x$.

g 5. If A , B , and C are the angles of a triangle, show that $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \cdot \sin B \cdot \sin C$.

g 6. Prove that $\frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} = \tan 60^\circ$.

g 7. The sum of the sides of a triangle is 100. The angle at A is double that at B , and the angle at B is double that at C . Find the sides of the triangle.

g 8. A pole is fixed on the top of a mound, and the angles of elevation of the top and bottom of the pole are 60° and 30° , respectively. Prove that the length of the pole is twice the height of the mound.

g 9. Two circles whose radii are R and r touch each other externally. If A is the angle included between the common tangents to the two circles, prove

$$\sin A = \frac{4(R-r)\sqrt{Rr}}{(R+r)^2}.$$

g 10. Find the angles satisfying this equation:

$$\sin^3 X - \sin X = \sin^3 18^\circ - \sin 18^\circ.$$

+ CALCULUS

h 1. A is known to be three times as truthful as B . What is the chance of the truth of an assertion which B affirms and A denies?

h 2. The axes of three mutually perpendicular right circular cylinders intersect in a common point. If the radii are equal, find the common volume.

h 3. Show that the volume inclosed by the surface $(x^2 + y^2 + z^2)^5 = (a^3x^2 + b^3y^2 + c^3z^2)^2$ is $\frac{4}{3}\pi(a^3 + b^3 + c^3)$.

h 4. Find the area of one loop of the curve $\rho = a \cos 2\theta$.

h 5. Differentiate the following

$$(a) \quad y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$$

$$(b) \quad y = \operatorname{arc} \sec \frac{x^2 + 1}{x^2 - 1}.$$

$$(c) \quad y = \log \frac{1 + \sqrt{x}}{1 - \sqrt{x}}.$$

h 6. Evaluate $\int_0^a \sqrt{a^2 - x^2} dx$.

h 7. Find the area inclosed by the curve $\rho = a \cos 5\theta$.

h 8. Find the pressure on one face of a submerged vertical equilateral triangle of side 4 feet, one side lying in the surface of the water.

h 9. Find the area bounded by the curves $x(y - e^x) = \sin x$, and $2xy = 2 \sin x + x^3$, the axis of y , and the ordinate $x = 1$.

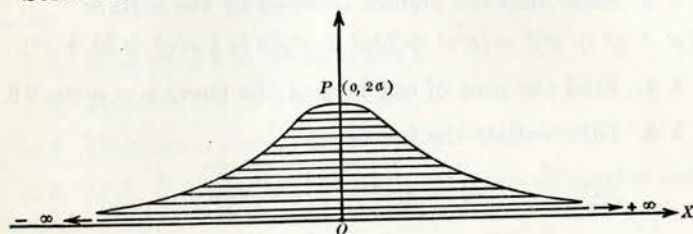
h 10. Find the volume generated by revolving the area bounded by the curve $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

h 11. Find the surface of the torus (ring) generated by revolving the circle $x^2 + (y - b)^2 = a^2$ about OX .

h 12. Find by double integration the area bounded by the curves $y^2 = x + 4$ and $y^2 = 4 - 2x$.

h 13. Find the area between the witch $y = \frac{8a^3}{x^2 + 4a^2}$, the x -axis and its asymptote.

Solution.



$$\begin{aligned} \text{Area of } OXP &= \int_0^{+\infty} y dx = \int_0^{+\infty} \frac{8a^3 dx}{x^2 + 4a^2} \\ &= \lim_{b \rightarrow \infty} \int_0^b \frac{8a^3 dx}{x^2 + 4a^2} = \lim_{b \rightarrow \infty} 8a^3 \int_0^b \frac{dx}{x^2 + 4a^2} \\ &= \lim_{b \rightarrow \infty} 8a^3 \left[\frac{1}{2a} \arctan \frac{x}{2a} + c \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left[4a^2 \arctan \frac{b}{2a} \right] = 4a^2 \cdot \frac{\pi}{2} = 2\pi a^2. \end{aligned}$$

$\therefore$ area between the witch and x -axis
 $= 2(2\pi a^2)$, or $4\pi a^2$.

h 14. Find the area within the hypocycloid $x = a \cos^3 \theta$,
 $y = a \sin^3 \theta$.

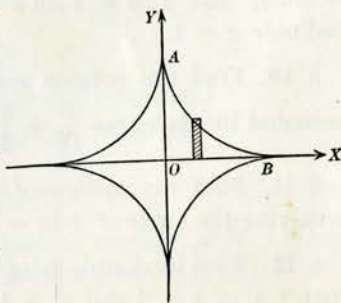
Solution. Area of AOB is
 one fourth of the required area.

$$\frac{A}{4} = \int_0^a y dx$$

$$= \int_0^{\pi/2} a \cos^3 \theta (3a \sin^2 \theta \cos \theta) d\theta.$$

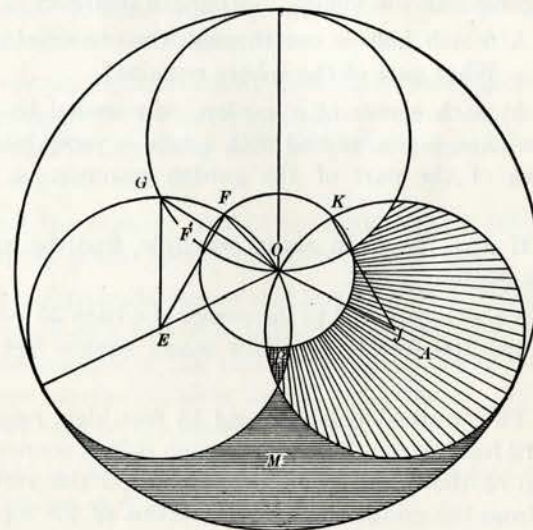
$$= 3a^2 \int_0^{\pi/2} \cos^4 \theta \sin^2 \theta d\theta = \frac{3}{8} \pi a^2.$$

$$\therefore A = \frac{3}{8} \pi a^2.$$



NUTS FOR THE DOCTOR

1. Within a garden, in the form of an equilateral triangle, is located a tree which is 20, 28, and 31 rods respectively from the corners. What is the side of the garden?



2. Find the areas of P , M , and A in the above diagram, given the area of the large circle 160 acres and the area of the small circle 1 acre.

3. A and B , starting from the same point, travel for 10 hours at their respective rates of 10 miles and 6 miles per hour. A travels northeast while B travels north. At the end of the 10 hours they turn and travel directly toward each other at their former rates until they meet. How far did each travel?

4. A hole d feet in diameter is bored through a wooden ball D feet in diameter. What is the volume removed?
5. If $D = 2$ feet and $d = 1$ foot, find the volume removed in example 4.
6. A hole is bored through the center of a sphere D inches in diameter. Find the diameter of the auger if m denotes the fractional part of the volume remaining.
7. A hole is bored through the center of an 8-inch ball consuming one half the volume. Find the diameter of the ball.
8. A 6-inch hole is cut through the center of a 10-inch sphere. What part of the sphere remains?
9. At each corner of a garden, surrounded by a wall n yards square, a goat is tied with a rope n yards long. Find the area of the part of the garden common to the four goats.
10. If $n = 10$ yd. in above example, find the area common to the four goats.
11. A cow is tethered to the corner of a barn 25 feet square, by a rope 100 feet long. How many square feet can she graze?
12. Two vertical poles 22 and 15 feet high, respectively, stand 12 feet apart. The foot of each pole is connected with the top of the other by ropes. Required the vertical distance from the ground to the intersection of the ropes.
13. A and B live at two opposite corners of a square lot; C and D live at the other two corners. They all carry water from a spring located in the lot. The spring is 5 rods from A, 4 rods from B, and 3 rods from C. How far must D carry water?
14. In a square garden stands a tree 30 feet, 40 feet, and 50 feet respectively from three successive corners. How much land is in the garden?

- + 15. Find the area over which a horse can graze, when tied by a rope 100 feet long to a stake in the circumference of a circular lake whose area is one acre.
- + 16. If the horse be tied outside the inclosure O' at O , find the area grazed over.
17. A piece of land is in the form of a right triangle. The sum of the sides about the right angle is 70 yards and their difference is equal to the length of a line, parallel to the shorter side, dividing the triangle into two equal parts. Find the length of the shorter side.
18. A certain number consists of 6 digits, units figure being 1. If this figure be removed and placed before the number, the new number resulting will be $\frac{1}{3}$ the original number. Find the original number.
19. Find the number of shot in a rectangular pile which has 600 in the lowest course and 11 in the top row.
20. A right triangle, base 8 and perpendicular 6, is divided into two parts by a line from the vertex to the base. A line joining the centers of the two inscribed circles of the two parts intersects the dividing line at right angles. Find the diameter of each of the inscribed circles.
21. What is the length of a line that cuts off $\frac{1}{3}$ of a circle which is 10 feet in diameter?
- + 22. A circular wall of radius a stands in a large field. A cow is tethered to the outside of this wall by a rope the length of which is equal to half the circumference of the wall. Find the area over which the cow can graze.
23. On the radii vectors of one loop of the lemniscate $\rho^2 = a^2 \cos 2\theta$ as diameters, circles are described passing through the pole. Find the locus of their points of intersection, and show that the area is twice that of the loop.

24. Find the length of the arc of the cissoid $\rho = 2a \tan \theta$ $\sin \theta$ from $\theta = 0$ to $\theta = \frac{\pi}{4}$.

25. An elliptical field has a major axis of 100 feet and a minor axis of 10 feet. A cow is tethered at the end of the major axis and another at the end of the minor axis. If each cow can graze over half the field, how long is the rope of each? What is the area of the portion over which the cows can graze in common?

26. The ellipse $\left(\frac{x^2}{81}\right) + \left(\frac{y^2}{16}\right) = 1$ is revolved around the y -axis. Find the area of the surface generated.

27. Show that the area inclosed by each of the following three curves is equal to the circle of radius a ; viz., πa^2 .

- (1) $a^2 x^2 = y^3(2a - y)$, (2) $a^2 - x^2 = (y - mx^2)^2$,
(3) $(xy + c + bx^2)^2 = x^2(a^2 - x^2)$.

28. Show that $\int_0^\infty e^{-x^2 - \frac{a^2}{x^2}} dx = \frac{\sqrt{\pi}}{2e^{2a}}$ by a transformation, rather than by the usual method of differentiating under the sign of integration:

29. Find the entire area of the curve
 $\rho = a(\sin 2\theta + \cos 2\theta)$.

30. Find the length of one turn of the logarithmic spiral $\rho = ae^{b\theta}$ from $\theta = 0$ to $\theta = \pi$.

31. Find the area of a loop of the trochoid
 $x = \frac{1}{3}a(3\phi - \pi \sin \phi)$, $y = \frac{1}{3}a(3 - \pi \cos \phi)$.

32. With the use of the compasses alone construct a circle with area five times as great as that of a given circle. (This problem is said to be due to Napoleon I.)

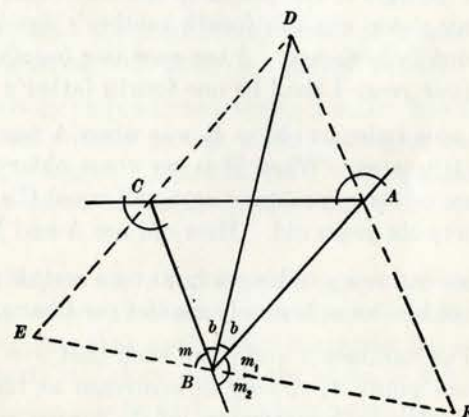
33. From a given point without a triangle to draw a line bisecting the triangle.

34. Construct the triangle if the three radii of the escribed circles are given.

35. Through four given points pass four lines which shall form a square.

36. By the compass alone find the mid-point of a line segment.

37. If the bisectors of the exterior angles of a triangle are produced in order, the line joining any vertex of this triangle to the opposite vertex of the original triangle is perpendicular to the side of the constructed triangle.



Solution.

Proof. $\angle m = \angle m_2 = \angle m_1$
 $\therefore \angle m = \angle m_1$

AD is the locus of a point equidistant from AB and AC .

Also DC is the locus of a point equidistant from CB and AC .

$\therefore D$ is equidistant from AB and BC and hence DB bisects $\angle CBA$.

Since $2m + 2b = 180^\circ$, $m + b = 90^\circ$.

$\therefore DB$ is $\perp$ to EF .

38. If $x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = 0$,
 prove that
 $[x^2 + y^2 + z^2 - 2(yz + zx + xy)]^2 = 128xyz(x + y + z)$.

+ 39. The logarithms (Briggian) of two numbers differ by 1.4238 and the numbers themselves by 3856. Find the numbers.

40. From a point without a square the distances to the three nearest vertices of the square are 20, 30, and 40. Find the side of the square.

41. Find the ages of the people in the statement: When I was born my sister was one fourth mother's age but she is now one third father's age. I am now one fourth mother's age but in four years I shall be one fourth father's age.

42. A is now twice as old as B was when A was six years older than B is now. When B is six years older than A is now, the sum of their combined ages will equal C's age then. C is now forty-six years old. How old are A and B?

43. A man can row y miles per hour on a certain river with the current in his favor, but only x miles per hour against the current. The numbers x and y are such that $x = \sqrt[y]{y} = y^{\frac{x}{y}}$. Starting from point A, rowing downstream as far as point B, and returning to his starting point, rowing continually at the rates above specified, he makes the round trip in $7\frac{1}{2}$ hours. Find the distance AB, numerically expressed.

+ 44. The logarithm of a number to the base 10 equals the sum of its logarithms to two other bases, of which one base is one tenth of the other base. What are the two bases?

45. It appears that on a certain day last week Mrs. A started out to call on Mrs. B, and when she reached the domicile of the latter learned that Mrs. B had been gone just

eleven minutes on her way to Mrs. A; and so Mrs. A at once returned to her home. Mrs. B had reached the mansion of Mrs. A fifteen minutes after the latter had left; and so she retraced her steps toward her own home. On their return trips the ladies met at a point just midway between their homes. How many minutes does it take each of them to walk from one house to the other?

46. When I was born, my sister was $\frac{1}{4}$ as old as my mother. She is now $\frac{1}{3}$ as old as father. In four years, I shall be $\frac{1}{4}$ as old as father. I am now $\frac{1}{4}$ as old as mother. Which two members of our family have the same birthday?

47. Prove that the sum of any series of consecutive odd numbers beginning with unity is a perfect square.

48. A railway embankment across a valley has the following dimensions: width on top, 20 ft.; width on base, 45 ft.; height, 11 ft.; length of top, 1020 yd.; length at base, 960 yd. Find the volume in cubic yards. Solve by prismatoid formula.

49. Four sailors and a monkey have spent the day gathering coconuts which they decide to divide equally the next day. It happens that each sailor mistrusts the others and each one steals out alone during the night to get his share. Each divides the pile he finds into four equal parts and takes his share away. In each case there is one odd coconut, which is given to the monkey. On the following day they meet and divide the remaining pile into four equal parts of which each one takes one part and again there is one left which they give to the monkey. How many coconuts in the original pile and how many does each sailor get?

50. Five men and a monkey were marooned on an island. They spent the first day in gathering coconuts. It happened that each sailor mistrusted the others, so each one

stole out alone during the night to get his share. Each divided the pile he found into five equal parts and hid his share. In each case there was one odd coconut which was given to the monkey. On the following day they met and divided the remaining pile into *exactly five equal* piles. How many coconuts were there in the original pile?

A KETTLE PROBLEM

- ✦ 51. One night I chanced with a tinker to sit,
Whose tongue ran a great deal faster than his wit.
He talked of his wit and an abundance of metal,
So I asked him to make me a flat-bottom kettle.
The top and the bottom must be
In just such proportion as five is to three;
Twelve inches in depth I proposed, and no more,
To hold in gallons seven less than a score.

HOW MUCH GOLD IS REQUIRED?

- ✦ 52. "This ball is iron within," said A,
"And gold without.; and now I pray
You, neighbor B, will you just say
How much the gold on it will weigh?"
The scales were brought and neighbor B
With careful, skillful hand did see
'Twas just an ounce. Then A said quick
"The iron in it is two feet thick;
And now this iron I'll straightway heat
And melt into a rod so neat,
Round, and six inches thick you see.
Now can you tell me, neighbor B,
Just how much of the gold 'twill take
To plate the rod that I shall make,
And have its thickness once for all
The very same 'twas on the ball?"

NUTS CRACKED FOR THE WEARY

ARITHMETIC

1. The longest line that can be stretched in a circular race track is 200 ft. Find area of track.

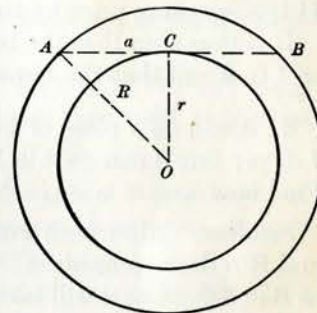
Solution. Let $AB = 2a$,
 $OA = R$, and $OC = r$.

Then

$$\text{area of track} = \pi(R^2 - r^2).$$

But in $\triangle ACO$, $R^2 - r^2 = a^2$.

Multiplying by π , we get,
 $\pi(R^2 - r^2) = \pi a^2$, the area required.



Substituting 100 for a , we have,

$$\text{Area} = 100^2 \pi = 31,415.9265 \text{ sq. ft.}$$

NOTE. Since πa^2 is the area of a circle whose radius is a , then the area of a circular ring or annulus equals that of a circle whose diameter is the length of the longest line that can be drawn in the ring.

2. Divide the twelfth power of 89,798 by the eleventh power of the same number and multiply the quotient by the number whose square root is equal to its cube root.

Solution. The twelfth power of any number divided by its eleventh power gives the number itself, and the only number whose square root equals its cube root is 1. Hence $89,798 \div 1 = 89,798$.

3. By discounting a note at 15% per annum, I get 20% interest per annum. How long does the note run?

Solution. Divide the difference of the rates expressed in hundredths by their product expressed in hundredths, and the quotient will be the time in years.

Thus $(.20 - .15) \div (.20 \times .15) = 1\frac{2}{3}$ years.

4. I sell an article for \$21, losing $12\frac{1}{2}\%$. What should I sell it for to gain $12\frac{1}{2}\%$?

Solution. Let 100% = the cost price.

Then $87\frac{1}{2}\%$ = selling price by the first condition and $112\frac{1}{2}\%$ = selling price by the second condition.

It is thus seen that the two selling prices are to each other as 7 to 9 and that the second is $\frac{9}{7}$ of the first.

5. A can do a piece of work in 10 days when B helps him 7 days; but B can do it in 12 days when A helps him 7 days. Find how long it takes each alone?

Solution. After both work 7 days, it would take A 3 days and B 5 days to finish it. Hence A in 3 days does as much as B in 5 days, or it will take him $\frac{3}{5}$ as long as B.

$\frac{3}{5}$ of $7 + 10$, or $14\frac{1}{5}$ days, is A's time.

$\frac{5}{3}$ of $7 + 12$, or $23\frac{2}{3}$ days, is B's time.

6. Find the smallest number which is divisible by 13, and which when divided by any of the numbers from 2 to 12 inclusive leaves a remainder of 1.

Solution. The L. C. M. of numbers 2 to 12 inclusive is 27,720. The required number is of the form $27,720n + 1$. Dividing by 13, we have,

$$\frac{27720n + 1}{13}, \text{ or } 2132n + \frac{4n + 1}{13}.$$

Now if the number is to be divisible by 13, then $4n + 1$ must be a multiple of 13. Hence $n = 3$ is obviously the smallest integral value of n to satisfy this condition.

$\therefore$ required number = $3 \times 27,720 + 1$, or 83,161.

7. Three cyclists, A, B, and C, ride at rates of 639 yd., 615 yd., and 603 yd. per minute, respectively. They start together in the same direction around a circular track, one quarter of a mile in circumference. How long after they start and at what part of the track will they next be together?

Solution. A gains 36 yd. per minute over C, or gains 440 yd. and meets him in $12\frac{2}{3}$ min.

B gains 12 yd. per minute over C, or gains 440 yd. and meets him in $18\frac{1}{3}$ min.

Then the three meet in $36\frac{2}{3}$ min. since $36\frac{2}{3}$ is the L. C. M. of $18\frac{1}{3}$ and $12\frac{2}{3}$.

In this time C goes $36\frac{2}{3} \times 603$ yd., or 22,110 yd., going around the course $50\frac{1}{4}$ times. Hence they meet at a point 110 yd. beyond the starting point.

8. A 25-gallon barrel is full of water. If 3 gallons are taken out each day and 2 gallons put in at night, in how many days will the barrel be empty?

Solution. Since the barrel contains one gallon less each day, and on the last day he will not put any in, it will take $25 - 2$, or 23 days.

9. A man at his marriage agreed that if at his death he should leave only a daughter, his wife should have $\frac{3}{4}$ of his estate; and if he should leave a son, she should have $\frac{1}{4}$. He left a son and a daughter. What fractional part of the estate should each receive, and how much was each one's portion, the estate being worth \$4000?

Solution. The wife was to have $\frac{3}{4}$, or 3 times as much as the daughter. The son was to have $\frac{3}{4}$, or 3 times as much as the wife. Then the daughter has 1 part, wife 3 parts, and son 9 parts.

$\therefore \frac{1}{13}$ of \$4000, or $\$307\frac{2}{13}$, is the daughter's share.

$\frac{3}{13}$ of 4000, or $923\frac{1}{13}$, is the wife's share.

$\frac{9}{13}$ of 4000, or $2769\frac{3}{13}$, is the son's share.

10. What are the three equal rates of trade discount equivalent to a single discount of 48.8%?

Solution. $100\% - 48.8\% = 51.2\% = .512$. $.512$ is the product of three equal factors, one of which is $\sqrt[3]{.512} = .8$, or 80%. The required rates are $100\% - 80\%$, or 20%.

11. I bought half a gross of marbles for as many cents as I got marbles for 8 cents. Find the price of the marbles.

Solution. If n be the number bought for 8 cents, then

$$72 : n = n : 8,$$

$$\text{or } n = 24.$$

$\therefore$ the price was 3 for a cent.

12. Find the least sum of money which can be paid in pence, shillings, florins, half-crowns, crowns, seven-shilling pieces, half-guineas, pounds, and guineas.

Solution. Obviously the sum is to be such as can be paid exactly in either of the various denominations above.

Expressing each in terms of pence, the lowest denomination, we have,

1, 12, 24, 30, 60, 84, 126, 240, 252.

The L. C. M. of these numbers is 5040.

The required amount is therefore 5040 pence, or £21.

13. Two clocks both indicate the true time to-day at noon. One gains a second an hour and the other loses three seconds in two hours. When next will they both indicate the same time, and when next will they both indicate the true time?

Solution. Let x = number of hours in which they will again indicate the same time. To indicate the same time, the time gained by the first plus the time lost by the second must equal 12 hours, or 43,200 seconds.

$$\text{Then } x + \frac{3}{2}x = 43,200$$

$$\therefore x = 17,280 \text{ hours, or } 720 \text{ days.}$$

They will both indicate true time when they have gained and lost 12 hours respectively. The first will gain 12 hours in 43,200 hours or 1800 days. The second will lose 12 hours in $\frac{2}{3}$ of 43,200 hours, or 1200 days. Hence the time required for the two clocks to both indicate the correct time together is the L. C. M. of 1200 and 1800, or 3600 days.

14. The beam of a defective balance is horizontal when the weight in one scale is $\frac{1}{4}$ more than the weight in the other. A merchant placing a pound weight alternately in the two scales of the balance sells what he thinks to be two pounds of cheese. How many pounds did the customer actually get?

Solution. When the cheese is placed in the side requiring extra weighting he sells $\frac{1}{4}$ of a pound, and when placed in the other side a weight of $\frac{1}{4}$ of a pound would be required to sell an exact pound of cheese. Hence if a pound weight is used only $\frac{1}{4}$ of a pound of cheese would be sold. Hence the amount of cheese sold = $\frac{1}{4}$ lb. + $\frac{1}{4}$ lb., or $2\frac{1}{2}$ lb.

15. A man stands on the circumference of a circular pond whose diameter is 13 rods, and he is 12 rods in a straight line from the farthest end of the diameter. How far is he from the nearest end?

Solution. Distance = $\sqrt{13^2 - 12^2}$, or 5 rd.

16. If we reverse the digits of a number of two figures, the difference between the two numbers is always divisible by 9. Why is this?

Solution. Any number of two digits is composed of 10 times ten's digit + 1 times unit's digit. Its reverse is 1 times ten's digit + 10 times unit's digit.

The difference between the numbers is always 9 times one digit less 9 times the other and is therefore divisible by 9.

17. In a certain game A can give B 1 point in 10, B can give C 1 point in 12, and C can give D 1 point in 15. How many can A give D in 1000?

Solution. The ratio of B's playing to that of A's is $\frac{9}{10}$; C's to B's, $\frac{11}{12}$; D's to C's $\frac{14}{15}$. Hence D's to A's is

$$\frac{14}{15} \text{ of } \frac{11}{12} \text{ of } \frac{9}{10} = \frac{77}{100}.$$

$$\frac{1000}{100} - \frac{77}{100} = \frac{23}{100}.$$

$\therefore$ A can give D $\frac{23}{100}$ of 1000, or 230 points.

18. Show that the difference between the two circumferences or boundaries of a circular race track is $2\pi a$, or twice the width of the track times π , a being the width of the track.

Solution. Let a = width of track and d = diameter of inner circle. Then diameter of outer circle is

$$d + a + a = d + 2a.$$

Then outer edge of track = $\pi(d + 2a)$, or $\pi d + 2\pi a$.

Inner edge of track = $\pi(d)$, or πd .

Subtracting, we have the formula.

Difference between two boundaries of track = $2\pi a$.

19. If a man 6 ft. tall could walk around the earth on the equator, how much farther would the top of his head move than his feet.

Solution. Using formula found in example 18, we have,

$$\text{Difference} = 2\pi a = 12\pi, \text{ or } 37.699 \text{ ft.}$$

20. A horse is 8 seconds longer in pacing on the outer edge of a circular race track $2\frac{6}{11}$ rods wide than on the inner track. In what time does he pace a mile ($\pi = 3\frac{1}{7}$)?

Solution. Using formula in example 18, we have, outer edge - inner edge = $2\pi a$.

Hence difference in distances = $2\pi a = 2 \times 3\frac{1}{7} \times 2\frac{6}{11}$, or 16 rods.

His speed per second = 16 rods $\div$ 8, or 2 rd. per second.

$320 \div 2 = 160$ sec. or 2 min. 40 sec. to the mile.

21. A person on a train observed that $2\frac{1}{4}$ times the number of spaces between the telegraph poles passed in a minute was the rate of the train in miles per hour. How far apart were the poles?

Solution. No. spaces passed in an hour : no. miles traveled per hour = $60 : 2\frac{1}{4}$.

$\therefore$ there are $26\frac{2}{3}$ spaces per mile, or the poles are 198 ft. apart.

22. A fly is in the center of the floor of a room 30 ft. long 20 ft. wide, and 12 ft. high. How far will it walk by taking the shortest path to one of the upper corners of the ceiling?

Solution. It has the choice of two paths. If we drop to the level of the floor one end of the room, the path would be the diagonal of a rectangle, or the hypotenuse of a right triangle whose legs are 10 and 27 ($12 + 15$). But if one side of the room is dropped to the plane of the floor, the path would be the hypotenuse of a right triangle whose legs are 15 and 22 (12 plus 10). The latter is the shorter route, which is $\sqrt{15^2 + 22^2}$, or 26.627 ft.

23. Two candles are of equal length. The one is consumed uniformly in 4 hours, the other in 5 hours. If they are lighted at the same time, when will one be three times as long as the other?

Solution. Let x = time in hours and let the length of the candle be unity.

$$\text{Then } 3(1 - \frac{1}{4}x) = 1 - \frac{1}{5}x.$$

$$\therefore x = 3\frac{7}{11} \text{ hours.}$$

NOTE. For a purely arithmetical solution to this problem see "Mathematical Wrinkles," page 165.

24. A goes 10 miles per day, B 12, C 16, D 24, and E 30. All start together January 1, in going around a circular island in the same direction, 56 miles around. When will they all meet?

Solution. When 56 is divided by the rates per day, the fractional results show the time required for each to make one circuit. The number of days that must elapse before they all make a complete round and meet together is the L. C. M. of these fractions.

(L. C. M. of numerators, or 56) $\div$ (G. C. D. of denominators, or 2) = 28.

$\therefore$ they again met together January 29.

25. A train 352 yards long, going 40 miles an hour, crosses a bridge in 50 seconds. Find the length of the bridge.

Solution. For the train to run one mile, or 1760 yards, would require $1\frac{1}{2}$ minutes, and since it takes the train 50 seconds, or $\frac{5}{6}$ of a minute, to cross the bridge, we have the proportion $1\frac{1}{2} : \frac{5}{6} = 1760 : (977\frac{7}{9})$. The train runs $977\frac{7}{9}$ yards in 50 seconds.

$\therefore$ length of bridge = $977\frac{7}{9} - 352$, or $625\frac{7}{9}$ yards.

26. A boy counted his marbles by twos, threes, fours, fives, and sixes and had one left each time. He then counted them by sevens and none were left. What is the least number he could have had?

Solution. The L. C. M. of 2, 3, 4, 5, and 6, plus 1, or 61 satisfies the first condition. To satisfy the second, we must have a multiple of 7 which is also a multiple of 60, plus 1. Since $61 \div 7$ leaves 5 and $60 \div 7$ leaves 4, we must add 60's to 61 until the sum of the remainders is a multiple of 7.

$5 + 4 + 4 + 4 + 4 = 21$, a multiple of 7.

$\therefore 61 + 4(60)$, or 301 is the number.

27. A steamer moves through 8° of longitude daily in plying to and fro across the Atlantic. How long is it from one noon to the next?

Solution. The rate of the ship is $\frac{1}{3}^\circ$ per hour, while that of the sun is 15° . When they both move west, the sun gains $14\frac{2}{3}^\circ$; but when the ship moves east, the sun gains $15\frac{1}{3}^\circ$.

Then since the sun must make a gain of 360° in each case, the time from noon to noon is

$360 \div 14\frac{2}{3}$, or $24\frac{6}{11}$ hr. for sailing west.

$360 \div 15\frac{1}{3}$, or $23\frac{1}{3}$ hr. for sailing east.

28. It is 1800 miles from A to B, and the Sunshine Special annihilates the distance in 50 hours. The train averages 30 miles an hour from A to X and 55 miles an hour from X to B. How far is X from B?

Solution. If the average for the entire distance were 30 miles an hour, the train would have run 1500 miles, leaving 300 miles to be made up by running 55 miles per hour or 25 miles an hour faster. $300 \div 25$, or 12 hours, is therefore the time required to go from X to B.

$\therefore$ distance from X to B = 12×55 mi. or 660 mi.

29. A runs a mile race with B and loses; had his speed been $\frac{1}{3}$ greater, he would have won by 22 yd. Find the ratio of A's speed to B's.

Solution. From the last condition, if A were going $1\frac{1}{3}$ times his rate, he would go 1760 yards while B goes $1760 - 22$, or 1738. But at A's ordinary rate he goes $\frac{3}{4}$ of 1760, or 1320 yards.

$\therefore$ ratio of their speeds is $1320 : 1738$, or $60 : 79$.

ALGEBRA

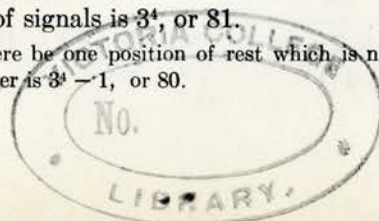
+1. Supposing that a man can place himself in 3 different attitudes, how many signals can be made by 4 men placed side by side?

Solution. The number of arrangements of n different elements, taking r at a time, is n^r .

Here $n = 3$ and $r = 4$.

$\therefore$ number of signals is 3^4 , or 81.

NOTE. If there be one position of rest which is not regarded as a signal, the number is $3^4 - 1$, or 80.



2. By drilling a hole through a cubical block of marble, parallel to the faces of the block, .0019635 part of the stone was cut away. What were the dimensions of the block?

Solution. Let x denote the edge of the cube and take the diameter of the hole unity, say one inch, then the volume of the hole is .7854 x and that of the cube is x^3 . Then

$$.0019635 x^3 = .7854 x.$$

or

$$x = 20.$$

3. If $\frac{b-c}{y-z} + \frac{c-a}{z-x} + \frac{a-b}{x-y} = 0$,

prove $(b-c)(y-z)^2 + (c-a)(z-x)^2 + (a-b)(x-y)^2 = 0$.

Solution. Let $p = x - y$, and $q = y - z$. Then $x - z = p + q$ and $\frac{b-c}{q} - \frac{c-a}{p+q} + \frac{a-b}{p} = 0$.

Clearing of fractions and rearranging,

$$p^2(b-c) - 2pq(c-a) + q^2(a-b) = 0.$$

Adding and subtracting $(p^2 + q^2)(c-a)$, we have

$$-q^2(b-c) - (p^2 + 2pq + q^2)(c-a) - p^2(a-b) = 0.$$

Changing signs and replacing x , y , and z , we have,

$$(b-c)(y-z)^2 + (c-a)(z-x)^2 + (a-b)(x-y)^2 = 0.$$

4. Simplify the expression :

$$\frac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1}.$$

$$\begin{aligned} \text{Solution. } \frac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1} &= \frac{\sqrt{3} \cdot \sqrt{4+2\sqrt{3}}}{\sqrt{3}+1} \\ &= \frac{\sqrt{3} \cdot \sqrt{3+2\sqrt{3}+1}}{\sqrt{3}+1} = \frac{\sqrt{3} \cdot \sqrt{(\sqrt{3}+1)^2}}{\sqrt{3}+1} = \sqrt{3}. \end{aligned}$$

5. Solve the equation, $2^x + 2^{x-1} = 10$.

Solution. Factoring,

$$2^x(1 + 2^{-1}) = 10.$$

$$\therefore 2^x = 10 \times \frac{2}{3}.$$

$$\therefore x \log 2 = \log 10 + \log 2 - \log 3.$$

$$= 1 - \log 3 + \log 2.$$

$$\therefore x = \frac{1 - \log 3}{\log 2} + 1 = 2.7369+.$$

6. Simplify $\frac{(4 + \sqrt{15})^{\frac{3}{2}} + (4 - \sqrt{15})^{\frac{3}{2}}}{(6 + \sqrt{35})^{\frac{3}{2}} - (6 - \sqrt{35})^{\frac{3}{2}}}$.

Solution. The expressions in parentheses become perfect squares when multiplied by $2^{\frac{3}{2}}$. Hence multiplying both numerator and denominator, we have,

$$\frac{(8 + 2\sqrt{15})^{\frac{3}{2}} + (8 - 2\sqrt{15})^{\frac{3}{2}}}{(12 + 2\sqrt{35})^{\frac{3}{2}} - (12 - 2\sqrt{35})^{\frac{3}{2}}} = \frac{(\sqrt{5} + \sqrt{3})^3 + (\sqrt{5} - \sqrt{3})^3}{(\sqrt{7} + \sqrt{5})^3 - (\sqrt{7} - \sqrt{5})^3}.$$

Expanding and collecting, we have

$$\frac{2[(\sqrt{3})^3 + 3\sqrt{5}(\sqrt{3})^2]}{2[3(\sqrt{7})^2\sqrt{5} + (\sqrt{5})^3]} = \frac{5\sqrt{5} + 9\sqrt{5}}{21\sqrt{5} + 5\sqrt{5}} = \frac{7}{13}.$$

7. Simplify, $\sqrt{7 + 2\sqrt{10}} - \sqrt{2} + \frac{3}{\sqrt{5}}$.

Solution.

$$\begin{aligned} \sqrt{7 + 2\sqrt{10}} - \sqrt{2} + \frac{3}{\sqrt{5}} &= \sqrt{2 + 2\sqrt{10} + 5} - \sqrt{2} + \frac{3}{\sqrt{5}} \\ &= \sqrt{(\sqrt{2} + \sqrt{5})^2} - \sqrt{2} + \frac{3}{\sqrt{5}} \\ &= \sqrt{2} + \sqrt{5} - \sqrt{2} + \frac{3}{\sqrt{5}} \\ &= \frac{8}{\sqrt{5}}. \end{aligned}$$

8. Simplify, $\sqrt{11 + 6\sqrt{2} + 4\sqrt{3} + 2\sqrt{6}}$.

Solution. Let the expression equal to $\sqrt{x} + \sqrt{y} + \sqrt{z}$.

Then $11 + 6\sqrt{2} + 4\sqrt{3} + 2\sqrt{6} = x + y + z + 2\sqrt{xy} + 2\sqrt{xz} + 2\sqrt{yz}$.

and $x + y + z = 11$,

$$\sqrt{xy} = 3\sqrt{2}$$

$$\sqrt{xz} = 2\sqrt{3}$$

$$\sqrt{yz} = \sqrt{6}.$$

Solving, $x = 6, y = 3, z = 2$.

$$\therefore \sqrt{11 + 6\sqrt{2} + 4\sqrt{3} + 2\sqrt{6}} = \sqrt{6} + \sqrt{3} + \sqrt{2}.$$

9. A man desires to purchase eggs at 5 cents, 1 cent, and $\frac{1}{2}$ cent, respectively, in such numbers that he will obtain 100 eggs for a dollar. How many solutions in rational integers?

Solutions. Let x, y , and z equal the number bought at 5 cents, 1 cent, and $\frac{1}{2}$ cent, respectively.

Then $x + y + z = 100$

and $5x + y + \frac{1}{2}z = 100$.

Eliminating z , we get the indeterminate equation:
 $9x + y = 100$, or $y = 100 - 9x$.

This gives the following integral solutions:

$$x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.$$

$$y = 91, 82, 73, 64, 55, 46, 37, 28, 19, 10, 1.$$

$$z = 8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88.$$

10. Solve $x^{\frac{5}{3}} + x^{\frac{2}{3}} = (2^8 + 2^{-8}) \times \sqrt[5]{x^{\frac{2}{3}}}$.

Solution. Transposing and factoring:

$$x^{\frac{2}{3}}(x^{\frac{5}{3}} - 2^8)(x^{\frac{2}{3}} - 2^{-8}) = 0.$$

$$\therefore x^{\frac{2}{3}} = 0, x^{\frac{5}{3}} = 2^8, x^{\frac{2}{3}} = 2^{-8}.$$

(a) Since $x^{\frac{3}{5}} = 0$, $\therefore x = 0$.

(b) Since $x^{\frac{8}{15}} = 2^8$, we have,

$$x^8 = 2^{8.15} \text{ and}$$

$$x^4 = + 2^{4.15} \text{ or } - 2^{4.15}.$$

$$\therefore x^2 = 2^{2.15}, - 2^{2.15} + i 2^{2.15}, - i 2^{2.15}.$$

$$\therefore x = \pm 2^{1.5}, \pm i 2^{1.5}, \pm i^{\frac{1}{2}} 2^{1.5}, \pm i^{\frac{3}{2}} 2^{1.5}.$$

(c) Since $x^{\frac{8}{15}} = 2^{-8}$, we may solve as above and find the values.

$$x = \pm 2^{-1.5}, \pm i 2^{-1.5}, \pm i^{\frac{1}{2}} 2^{-1.5}, \pm i^{\frac{3}{2}} 2^{-1.5}.$$

$$\therefore x = 0, \pm 2^{\pm 1.5}, \pm i 2^{\pm 1.5}, \pm i^{\frac{1}{2}} 2^{\pm 1.5}, \pm i^{\frac{3}{2}} 2^{\pm 1.5}.$$

11. Solve $10x^4 - 53x^3 + 37x^2 + 53x + 10 = 0$.

Rearranging terms and dividing by x^2 ,

$$10\left(x^2 + \frac{1}{x^2}\right) - 53\left(x - \frac{1}{x}\right) + 37 = 0.$$

$$10\left(x^2 - 2 + \frac{1}{x^2}\right) - 53\left(x - \frac{1}{x}\right) + 57 = 0.$$

$$10\left(x - \frac{1}{x}\right)^2 - 53\left(x - \frac{1}{x}\right) + 57 = 0.$$

$$\left[5\left(x - \frac{1}{x}\right) - 19\right]\left[2\left(x - \frac{1}{x}\right) - 3\right] = 0,$$

or $5x^2 - 19x - 5 = 0$ and $2x^2 - 3x - 2 = 0$.

Solving, $x = 2, -\frac{1}{2}, \frac{1}{10}(19 \pm \sqrt{461})$.

12. There are two numbers, each consisting of two digits, and the digits of the second are those of the first but reversed; the product of these numbers is 8722; if the first number be divided by the second, the quotient will be 1 with a remainder consisting of one figure only. Find the numbers.

Solution. Let x and y denote the tens' and units' digits of the first number respectively.

Then $(10x + y)(10y + x) = 8722$.

If $(10x + y)$ be divided by $(x + 10y)$, then the remainder, $9x - 9y$, or $9(x - y)$, is an integer less than 10. Therefore $(x - y) = 1$, and substituting $(y + 1)$ for x , we have

$$(11y + 10)(11y + 1) = 8722$$

$$\text{or } y^2 + y - 72 = 0.$$

$$\therefore x = 9, \text{ or } 8.$$

$\therefore$ the required number = 98, or 89.

13. A father's age is equal to those of his three children together. In 9 years it will amount to those of the two eldest; in 3 years after that, to those of the eldest and youngest; and in 3 years after that, to those of the two youngest. Find their present ages.

Solution. If x , y , and z denote the children's ages, then

$$x + y + z + 9 = (x + 9) + (z + 9).$$

$$x + y + z + 12 = (x + 12) + (z + 12).$$

$$x + y + z + 15 = (x + 15) + (y + 15).$$

Solving, $x = 9$, $y = 12$, $z = 15$.

The father's age is 36.

14. A is three times as old as B was when A was as old as B is now; and in 24 years' time B will be twice as old as A was when B was half as old as A is now. Find the ages of A and B.

Solution. Let $x = A$'s age and $y = B$'s age. $(x - y)$ years ago A was B's present age, and at that time B's age was $y - (x - y)$, or $2y - x$ years.

Then $x = 3(2y - x)$, or $x = \frac{3}{2}y$. 24 years hence B's age will be $y + 24$. The present age of A is x . It has been $(y - \frac{x}{2})$ years since B was that age and therefore at that time A's age was $x - (y - \frac{x}{2})$, or $\frac{3x}{2} - y$.

$$\text{Hence } y + 24 = 2\left(\frac{3x}{2} - y\right).$$

Substituting for x its value $\frac{3}{2}y$ and solving, we find $y = 16$ and $x = 24$.

15. If $a^2 + b^2 = 7ab$, prove that

$$\log \frac{1}{3}(a + b) = \frac{1}{2}(\log a + \log b).$$

Solution. Adding $2ab$ to each side of the equation, we have

$$(a + b)^2 = 9ab.$$

$$\therefore 2 \log(a + b) = 2 \log 3 + \log a + \log b.$$

$$\therefore \log(a + b) - \log 3 = \frac{1}{2}(\log a + \log b).$$

$$\therefore \log \frac{1}{3}(a + b) = \frac{1}{2}(\log a + \log b).$$

16. Solve $(a^4 - 2a^2b^2 + b^4)^{x-1} = (a - b)^{2x}(a + b)^{-2}$.

Solution. This may be written as follows:

$$[a^2 - b^2]^{2(x-1)} = [a - b]^{2x}[a + b]^{-2}.$$

$$\therefore (2x - 2)[\log(a + b) + \log(a - b)]$$

$$= 2x \log(a - b) - 2 \log(a + b).$$

$$\therefore 2x \log(a + b) = 2 \log(a - b).$$

$$\therefore x = \frac{\log(a - b)}{\log(a + b)}.$$

17. Simplify, $\frac{n^3 - 3n + (n^2 - 1)\sqrt{n^2 - 4} - 2}{n^3 - 3n + (n^2 - 1)\sqrt{n^2 - 4} + 2}$.

Solution.

$$\frac{n^3 - 3n + (n^2 - 1)\sqrt{n^2 - 4} - 2}{n^3 - 3n + (n^2 - 1)\sqrt{n^2 - 4} + 2}$$

$$= \frac{n^3 - 3n - 2 + (n^2 - 1)\sqrt{n^2 - 4}}{n^3 - 3n + 2 + (n^2 - 1)\sqrt{n^2 - 4}}$$

$$= \frac{(n^2 - n - 2)(n + 1) + (n^2 - 1)\sqrt{n^2 - 4}}{(n^2 + n - 2)(n - 1) + (n^2 - 1)\sqrt{n^2 - 4}}$$

$$= \frac{n + 1}{n - 1} \cdot \frac{[n^2 - n - 2 + (n - 1)\sqrt{n^2 - 4}]}{[n^2 + n - 2 + (n + 1)\sqrt{n^2 - 4}]}$$

$$= \frac{n + 1}{n - 1} \cdot \frac{[(n - 2)(n + 1) + (n - 1)\sqrt{n^2 - 4}]}{[(n + 2)(n - 1) + (n + 1)\sqrt{n^2 - 4}]}$$

$$= \frac{(n + 1)\sqrt{n - 2}}{(n - 1)\sqrt{n + 2}} \cdot \frac{[(n + 1)\sqrt{n - 2} + (n - 1)\sqrt{n + 2}]}{[(n - 1)\sqrt{n + 2} + (n + 1)\sqrt{n - 2}]}$$

$$= \frac{(n + 1)\sqrt{n - 2}}{(n - 1)\sqrt{n + 2}}.$$

$$(n - 1)\sqrt{n + 2}$$

18. Solve the equation, $x^3 + 16x = 455$.

Solution.

$$\begin{aligned} x^4 + 16x^2 &= 455x = 65 \times 7x. \\ x^4 + 65x^2 + \left(\frac{6.5}{2}\right)^2 &= 49x^2 + 65 \times 7x + \left(\frac{6.5}{2}\right)^2. \\ \therefore x^2 + \frac{6.5}{2} &= 7x + \frac{6.5}{2}. \\ \therefore x^2 &= 7x. \\ \therefore x &= 7. \end{aligned}$$

19. In what scale does 2t7 represent the number 475 in the scale of ten?

Solution. Let r = the radix of the scale.

Then $2r^2 + 10r + 7 = 475$.

Solving, $r = 13$.

20. Transform .5421 from the scale of 10 to the scale of 6.

Solution. This may be changed to the scale of 6 as follows:

$$\text{In the decimal scale, } .5421 = \frac{5}{10} + \frac{4}{10^2} + \frac{2}{10^3} + \frac{1}{10^4}.$$

$$\text{In the scale with radix } r, .5421 = \frac{5}{r} + \frac{4}{r^2} + \frac{2}{r^3} + \frac{1}{r^4}.$$

$$\begin{aligned} \text{In the scale of 6, } .5421 &= \frac{5}{6} + \frac{4}{6^2} + \frac{2}{6^3} + \frac{1}{6^4} \\ &= \frac{5 \cdot 6^3 + 4 \cdot 6^2 + 2 \cdot 6 + 1}{64} = \frac{1237}{1296} = .9545 \dots \end{aligned}$$

21. Find the cubic with rational coefficients of which $x = \sqrt[3]{2} + \sqrt[3]{4}$ is a root.

Solution.

If $x = \sqrt[3]{2} + \sqrt[3]{4}$, then

$$\begin{aligned} x^3 &= \sqrt[3]{8} + 3\sqrt[3]{4} \cdot \sqrt[3]{4} + 3\sqrt[3]{2} \cdot \sqrt[3]{16} + \sqrt[3]{64} \\ &= 6\sqrt[3]{2} + 6\sqrt[3]{4} + 6 = 6(\sqrt[3]{2} + \sqrt[3]{4}) + 6 = 6x + 6. \end{aligned}$$

22. Prove that $4^{3x+1} + 2^{3x+1} + 1$ is divisible by 7 for integral values of x .

Solution.

$$\text{Let } f(x) = 4^{3x+1} + 2^{3x+1} + 1 = 2^{6x+2} + 2^{3x+1} + 1.$$

$$\text{Then } f(x+1) = 2^{6x+8} + 2^{3x+4} + 1$$

$$\begin{aligned} \text{and } f(x+1) - f(x) &= 2^{6x+8} - 2^{6x+2} + 2^{3x+4} - 2^{3x+1} \\ &= (2^6 - 1)2^{6x+2} + (2^3 - 1)2^{3x+1}, \end{aligned}$$

which is a multiple of 7 since $(2^6 - 1) = 63$ and $2^3 - 1 = 7$. Hence if $f(x)$ is divisible by 7, so also is $f(x+1)$. By trial we find that $f(x)$ is divisible by 7 when $x = 1$. Hence by mathematical induction $f(x+1)$ is divisible by 7.

23. A contractor has to remove 2000 tons of rock in 22 days. His steam shovel can remove 63 tons a day. He can rent for \$50 a day another shovel which can remove 126 tons a day, but the excavation is narrow so that he can use only one shovel at a time. How much does he pay in rent for the larger shovel if he uses his own shovel as much as possible?

Solution.

Let x = no. days he uses own shovel.

Then $22 - x$ = no. days he uses hired shovel.

$63x$ = no. tons moved with own shovel.

$126(22 - x)$ = no. tons moved with hired shovel.

$$\therefore 63x + 2772 - 126x = 2000.$$

$$\therefore x = 12\frac{1}{3} \text{ and } 22 - x = 9\frac{2}{3}.$$

$$\therefore \text{rent due for hired shovel} = 9\frac{2}{3} \times \$50, \text{ or } \$487.30.$$

24. Eliminate x in the equations

$$x + \frac{1}{x} = y,$$

$$x^5 + \frac{1}{x^5} = z.$$

Solution.

$$\begin{aligned} y^5 - z &= \left(x + \frac{1}{x}\right)^5 - x^5 - \frac{1}{x^5} = 5\left[\left(x^3 + \frac{1}{x^3}\right) + 2\left(x + \frac{1}{x}\right)\right] \\ &= 5y\left(x^2 - 1 + \frac{1}{x^2} + 2\right) = 5y(y^2 - 1) = 5y^3 - 5y. \end{aligned}$$

$$\therefore z = y^5 - 5y^3 + 5y, \text{ is the eliminant.}$$

25. 12 oxen are turned into a pasture of $3\frac{1}{3}$ acres and eat all the grass in 4 weeks so that the pasture is bare. 21 oxen are turned into a pasture of 10 acres and eat all the grass in 9 weeks. How many oxen will eat all the grass of 24 acres in just exactly 18 weeks, it being assumed that the grass in all the pastures is at the same height when the oxen are turned in, and that the grass grows at a uniform rate.

Solution. Let x = number of pounds of grass each ox eats per week.

y = number of pounds of grass on each acre at first.

z = number of pounds of grass that grows per week per acre.

k = number of oxen required in the last condition.

Then $\frac{10}{3}y + \frac{40}{3}z = 48x$,

$10y + 90z = 189x$,

$24y + 432z = 18kx$.

Eliminating y , we have $10z = 9x$,

and $560z = (30k - 576)x$.

Solving, $k = 36$, the required no. of oxen.

26. Jones settled an annuity upon his three daughters to be divided in the same ratio as their ages. At the first payment the eldest was entitled to one half the annuity. When the sixth payment was made, Martha received one dollar less than she had the first year, Phoebe one seventh less than the first year, and Mary twice as much. What was the amount of the annuity?

Solution. Let a , b , and $a + b$ represent the ages in order, when the first annuity is paid, and let x equal the amount by which each age is multiplied to get the share of each of the annuity.

Then the share of each is ax , bx , and $(a + b)x$, and the total annuity is $2x(a + b)$.

When the sixth payment is made, each is five years older, or $a + 5$, $b + 5$, and $a + b + 5$.

Hence we have the equations

$$\frac{(a + 5)2x(a + b)}{2a + 2b + 15} = 2ax, \quad (1)$$

$$\frac{(b + 5)2x(a + b)}{2a + 2b + 15} = bx - 1, \quad (2)$$

$$\frac{(a + b + 5)2x(a + b)}{2a + 2b + 15} = \frac{6(a + b)x}{7}. \quad (3)$$

Eliminating x from [1] and [3], we have

$$7a + 7b + 35 = 6a + 6b + 45$$

$$\text{or} \quad a + b = 10. \quad (4)$$

From [1], we have

$$a^2 + 5a + ab + 5b = 2a^2 + 2ab + 15a$$

$$\text{or} \quad a^2 + 10a + ab = 5b. \quad (5)$$

Solving [4] and [5] for a and b , we get $a = 2$ and $b = 8$.

Substituting these values in [2], we find $x = 1\frac{3}{4}$.

$\therefore ax + bx + (a + b)x = \35 , the amount of the annuity.

27. How shall we buy twelve eggs for eighty cents, if hen eggs sell at five cents each, duck eggs at seven cents each, and turkey eggs at eight cents each, and if we buy some of each?

Solution. Let x , y , z , be the number of eggs of each kind, respectively.

Then $x + y + z = 12$

and $5x + 7y + 8z = 80$.

Eliminating x and solving for y , we have

$$y = 10 - \frac{3z}{2}.$$

Since y is an integer, this value of y , though fractional in form, must be an integer. Therefore, z is even. Assigning

even values to z , beginning with 2, we get three sets of answers satisfying the conditions of the problem as follows:

z	2	4	6
y	7	4	1
x	3	4	5

28. Find the value of x in the equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$.

Solution. Taking the logarithm of each member, we obtain

$$x\sqrt{x} \log x = x \log (x\sqrt{x}) = x \log (x)^{\frac{3}{2}} = \frac{3}{2} x \log x$$

$$\text{or } x(\sqrt{x} - \frac{3}{2}) \log x = 0.$$

$$\therefore x = 0, \log x = 0, \text{ and } \sqrt{x} - \frac{3}{2} = 0.$$

$$\therefore x = 0, x = 1, \text{ and } x = \frac{9}{4}.$$

29. Solve for x and y the equations

$$2^{x+y} = 6,$$

$$2^{x+1} = 3^y.$$

Solution. Taking logarithms, we have

$$(x + y) \log 2 = \log 6$$

$$\text{and } (x + 1) \log 2 = y \log 3.$$

Solving for x and y , we have

$$x = \frac{(\log 6)^2 - \log 2 \cdot \log 12}{\log 2 \log 6} = 1.198$$

$$\text{and } y = \frac{\log 12}{\log 2} = 1.387.$$

30. A and B start from the same place at the same time to travel around a circle of 10 miles in opposite directions. B goes more slowly than A until they meet, then by doubling his rate he next meets A at the starting point. Find where they first met.

Solution. Let x = A's rate, y = B's rate, and z = distance from starting point, when they first met.

$$\text{Then } \frac{10 - z}{x} = \frac{z}{y}, \text{ or } y = \frac{xz}{10 - z},$$

$$\text{and } \frac{10 - z}{2y} = \frac{z}{x}, \text{ or } y = \frac{x(10 - z)}{2z}.$$

$$\therefore xz(10 - z) = \frac{x(10 - z)}{2z}.$$

$$\therefore z = 10(\sqrt{2} - 1) = 4.142 \text{ mi.}$$

31. Solve the simultaneous equations,

$$x^2 + xy + y^2 = a,$$

$$x^4 + x^2y^2 + y^4 = b.$$

Solution. The equations may be written as follows:

$$x^2 + xy + y^2 = a, \quad (1)$$

$$(x^2 + xy + y^2)(x^2 - xy + y^2) = b. \quad (2)$$

$$\therefore x^2 - xy + y^2 = \frac{b}{a}. \quad (3)$$

From (1) and (3) we get

$$x^2 + y^2 = \frac{a^2 + b}{2a} \text{ and } 2xy = \frac{2a^2 - b}{2a}.$$

Solving for $x + y$ and $x - y$, we have

$$x + y = \pm \sqrt{\frac{3a^2 - b}{2a}} \text{ and } x - y = \pm \sqrt{\frac{3b - 2a}{2a}}.$$

$$\therefore x = \frac{1}{4a} [\pm \sqrt{6a^3 - 2ab} \pm \sqrt{6ab - 2a^3}]$$

$$\text{and } y = \frac{1}{4a} [\pm \sqrt{6a^3 - 2ab} \mp \sqrt{6ab - 2a^3}].$$

32. Divisor = multiplier; quotient = product; and multiplicand = half the dividend. Find the square of the divisor.

Solution. Let a = divisor or multiplier,

b = multiplier or half the dividend.

Then the quotient is $2b \div a$ and the product is ab .

$$\therefore \frac{2b}{a} = ab, \text{ or } a^2 = 2.$$

33. A, B, C, and D travel on a circular track at uniform rates. A and B are together in 11 hours; B and C in 7 hours; C and D in 12 hours; D and A in 8 hours; A and C in 10 hours. When will B and D be together?

Solution. Let x , y , and z denote A's hourly gains on B, C, D, respectively.

Then $y - x$, $z - x$, $z - y$ will denote B's hourly gains on C and D, and C's hourly gain on D, respectively.

If B and D are together in f hours, we have

$$11x + 7(y - x) = 10y, \quad (1)$$

$$10y + 12(z - y) = 8z, \quad (2)$$

$$8z + f(x - z) = 11x, \quad (3)$$

which may be written thus,

$$\frac{x}{y} = \frac{3}{4},$$

$$\frac{y}{z} = 2,$$

$$\frac{z}{x} = \frac{f - 11}{f - 8}.$$

Multiplying the above equations gives

$$1 = \left(\frac{3}{4}\right)(2)\left(\frac{f - 11}{f - 8}\right).$$

$$\therefore f = 17 \text{ hours.}$$

34. In how many ways can a sum of \$2.40 be made up of nickels, dimes, and quarters, on the condition that the number of nickels used shall equal the number of quarters and dimes together? Determine the various groups.

Solution. Let

n = the number of nickels,

d = the number of dimes,

q = the number of quarters.

and

$$\text{Then } 5n + 10d + 25q = 240 \quad (1)$$

and

$$n = d + q. \quad (2)$$

Eliminating n and solving for q , we have

$$d = 16 - 2q. \quad (3)$$

Now if q is any integer whatever, d has an integral value. Therefore the values which satisfy (3) are as follows:

q	1	2	3	4	5	6	7
d	14	12	10	8	6	4	2

Substituting these values in (2) gives the following respective values for n :

n	15	14	13	12	11	10	9
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Hence there are seven possible groups, all different.

35. How many calves at \$3.50, sheep at \$1.50, and lambs at \$.50 per head can be bought for \$100, the total number bought being 100?

Solution. Let c , s , and l represent the number of calves, sheep, and lambs respectively.

$$\text{Then } 3\frac{1}{2}c + 1\frac{1}{2}s + \frac{1}{2}l = 100 \quad (1)$$

$$\text{and } c + s + l = 100. \quad (2)$$

Eliminating l and solving for c , we have

$$c = \frac{100 - 2s}{6}. \quad (3)$$

Now if c be integral, $\frac{100 - 2s}{6}$ must be integral; that is,

$100 - 2s$ must be an integral multiple of 6.

Placing $100 - 2s$, equal to 6, 12, 18, ... 96 respectively, we obtain the following values of c and s which satisfy (3):

s	47	44	41	38	35	32	29	26	23	20	17	14	11	8	5	2
c	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16

Substituting these values in (2) gives the following respective values for l :

l	52	54	56	58	60	62	64	66	68	70	72	74	76	78	80	82
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PLANE GEOMETRY

1. If the apothem of a hexagonal lot be increased 4 feet, the area will be increased 400 square feet. Find the area of the lot.

Solution.

Let A = area of circumscribed regular triangle.

A' = area of regular inscribed triangle.

$OF = h$ and $OB = r$.

Since $\triangle OAB$ is equilateral,

$$h = \frac{1}{2} r\sqrt{3},$$

or $r = \frac{2}{3} h\sqrt{3},$

and $A' = \frac{3}{2} r^2\sqrt{3},$ or $2 h^2\sqrt{3}.$

$$A = 2(h + 4)^2\sqrt{3}.$$

$$\therefore A - A' = 2(h + 4)^2\sqrt{3} - 2 h^2\sqrt{3} = 400.$$

Solving, $h = (25 + \sqrt{3}) - 2.$

$$\begin{aligned} \therefore A' &= 2 h^2\sqrt{3} = 2(25 + \sqrt{3})^2\sqrt{3} \\ &= 2\left[\frac{637}{3} - (100 \div \sqrt{3})\right]\sqrt{3} = 535.5344 \text{ sq. ft.} \end{aligned}$$

2. One side of a field in the shape of a trapezoid is 22 rods and the other three sides are all equal. Find the length of each of the unknown sides if each diagonal is 25 rods.

Solution.

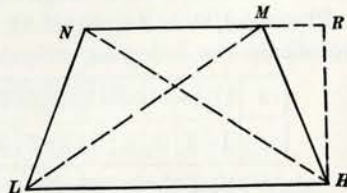
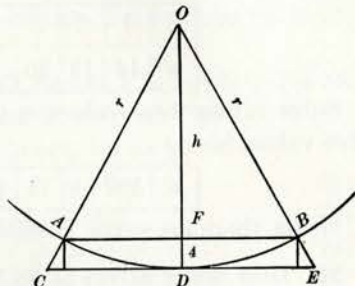
Given $LH = 22$ rods

and $LM = HN = 25$ rods.

Also $LN = NM = HM$
 $= 2x.$

Draw $HR \perp$ to NM produced.

Then $MR = \frac{1}{2}(22 - 2x) = 11 - x.$
 $NR = 11 - x + 2x = 11 + x.$



In rt. $\triangle HNR$ and HMR we have

$$\overline{HN}^2 - \overline{NR}^2 = \overline{HR}^2 \text{ and } \overline{HM}^2 - \overline{MR}^2 = \overline{HR}^2.$$

$$\therefore \overline{HN}^2 - \overline{NR}^2 = \overline{HM}^2 - \overline{MR}^2$$

or $25^2 - (11 + x)^2 = 4x^2 - (11 - x)^2.$

$$625 - 121 - 22x - x^2 = 4x^2 - 121 + 22x - x^2.$$

$$4x^2 + 44x = 625.$$

Completing the square and solving, $x = 8.156.$

$$\therefore 2x = LN = NM = HM = 16.312.$$

3. Two poles perpendicular to the same plane are a ft. and b ft. in height. At what height from the plane will lines drawn from the top of each to the base of the other cross?

Solution. By similar $\triangle$

$$a : c = x : y,$$

or $ay = cx. \quad (1)$

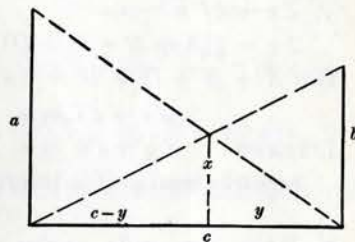
Also $b : c = x : c - x,$

or $bc - by = cx. \quad (2)$

Solving (1) and (2) for $y,$

we have

$$y = \frac{cx}{a} \text{ and } y = \frac{bc - cx}{b}.$$



Solving for $x,$ we have $x = \frac{ab}{a + b}.$

NOTE. The result is independent of the distance the poles are apart.

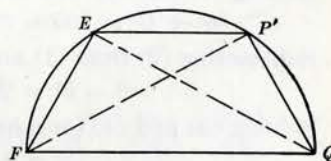
4. Three chords, lengths 6, 8, and 10, just go around in a semicircle. Find the radius of the circle.

Solution.

Let x = diameter.

Now $FP \cdot EQ = PQ \cdot PE$
 $+ PE \cdot FQ.$

$\triangle QEF$ and FPQ are rt. triangles.



Then

$$FP = \sqrt{x^2 - 36} \text{ and } EQ = \sqrt{x^2 - 100}.$$

$$\therefore (x^2 - 36)^{\frac{1}{2}}(x^2 - 100)^{\frac{1}{2}} = 60 + 8x.$$

$$\therefore x = 16.1116 \text{ and } r = 8.0558.$$

5. Prove that the four lines bisecting the angles of any quadrilateral form a quadrilateral which may be inscribed in a circle.

Solution.

$$\angle e = \pi - \frac{1}{2}(A + B).$$

$$\angle f = \pi - \frac{1}{2}(C + D).$$

$$\therefore \angle e + \angle f =$$

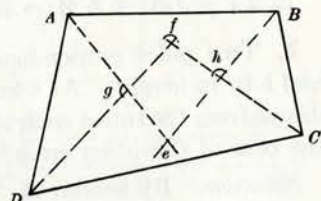
$$2\pi - \frac{1}{2}(A + B + C + D).$$

$$\text{But } A + B + C + D = 2\pi.$$

$$\therefore \angle e + \angle f = \pi.$$

$$\text{Likewise } \angle g + \angle h = \pi.$$

$\therefore$ quadrilateral $gehf$ is inscribable.



6. From the acute angles of a right triangle lines are drawn to the mid-points of the opposite sides; if their respective lengths are $\sqrt{73}$ and $\sqrt{52}$, find the sides of the triangle.

Solution. Let $BC = a$; $AC = b$;

$$AB = c; AD = \sqrt{52}$$

and

$$BE = \sqrt{73}.$$

$$\text{Then } a^2 + \frac{1}{4}b^2 = 73, \quad (1)$$

$$\frac{1}{4}a^2 + b^2 = 52. \quad (2)$$

Adding (1) and (2) and multiplying by $\frac{4}{3}$, we obtain

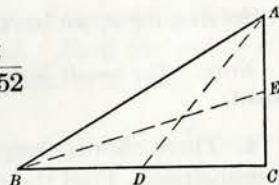
$$a^2 + b^2 = 100 = c^2. \quad (3)$$

Subtracting (2) from (1) and multiplying by $\frac{4}{3}$, we have

$$a^2 - b^2 = 28. \quad (4)$$

Solving (3) and (4) for a and b , we find

$$a = 8, b = 6, c = 10.$$



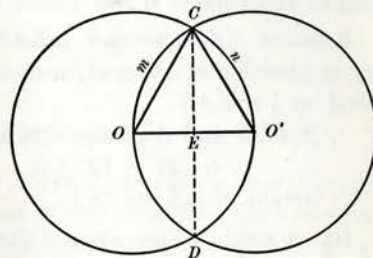
7. Two circles 20 ft. in diameter are placed in such a way that the circumference of each just touches the center of the other. Find the area of the space common to each circle.

Solution.

$\triangle OO'C$ is equilateral and its area $= 25\sqrt{3}$.

$$\text{Area of sector } O' - OmC = \frac{1}{6}(\pi 10^2) = \frac{100}{6}\pi.$$

$$\text{Area of segment } OmC \text{ or } O'nC = \frac{100\pi}{6} - 25\sqrt{3}.$$



$$\therefore \text{area } O'O m C n O' = 25\sqrt{3} + 2\left(\frac{100\pi}{6} - 25\sqrt{3}\right)$$

$$= \frac{100\pi}{3} - 25\sqrt{3} = 61.418.$$

$$\therefore \text{area common} = 2 \text{ times } 61.418 \text{ sq. ft. or } 122.836 \text{ sq. ft.}$$

8. If a, b, c denote the sides of a plane triangle, and R the radius of the circumscribing circle, show that the area of the triangle is $\frac{abc}{4R}$.

Solution. Given $\triangle ABC$, BD an altitude, and BE diameter.

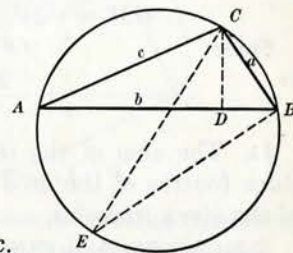
Then $\triangle ADC$ and CBE are similar

$$\text{and } AC : CE = CD : CB.$$

$$\therefore CE \cdot CD = AC \cdot CB = ac.$$

$$\therefore CD = \frac{ac}{CE} = \frac{ac}{2R}.$$

$$\therefore \text{area } \triangle ABC = \frac{1}{2}(AB \cdot CD) = \frac{1}{2}\left(b \cdot \frac{ac}{2R}\right) = \frac{abc}{4R}.$$



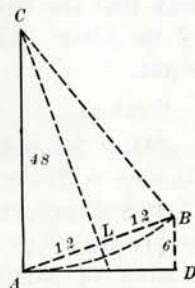
9. Find the height of a flag pole whose rope just reaches to the ground, but when pulled out 24 feet from the base of the pole and kept taut, the end of the rope is 6 feet above the plane.

Solution. $\angle ACL$ and $\angle BAD$, formed by a chord and tangent, are each measured by $\frac{1}{2}$ arc AB .

$\therefore \triangle ADB$ and ACL are similar.

$$\therefore 6 : 24 = 12 : CA.$$

$\therefore$ height of pole = 48 ft.



10. A circle whose area is 256π sq. ft. is described upon the perpendicular or leg of a right triangle as a diameter. From the point where the circle cuts the hypotenuse a tangent to the circle is drawn, which cuts the base. If the shortest distance from the point of intersection of the tangent with the base to the perpendicular is 18 feet, what is the length of the hypotenuse?

Solution. Since area of circle = 256π ,
 $OB = \sqrt{256}$,
 or 24 ft.

$$MC = 18 \text{ ft.}$$

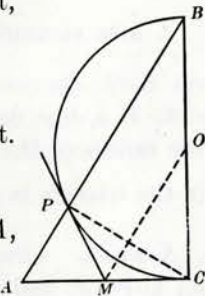
$$\therefore OM = \sqrt{24^2 + 18^2} = 30.$$

Since
 or

$$OC : BC = OM : BA,$$

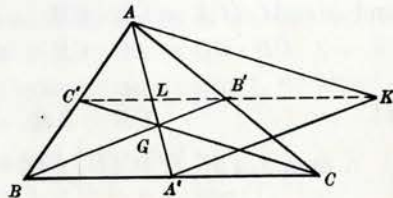
$$24 : 48 = 30 : BA,$$

$$\therefore BA = 60 \text{ ft.}$$



11. The area of the triangle of the medians is equal to three fourths of the area of the given triangles.

Solution. Produce $C'B'$ to K , making $B'K = B'C'$. Since the diagonals of $AKCC'$ bisect each other, it is a parallelogram and



$AK = CC' = m_c$. Also $A'BB'K$ is a parallelogram since $A'B$ is equal and parallel to $B'K$. Hence $A'K = BB' = m_b$. Thus $\triangle AA'K$ has for its sides the given medians and may therefore be constructed.

Area of $\triangle AA'K$ = area $\triangle AKL$ + area $\triangle A'KL$.

These $\triangle$ have the same base KL and their altitudes are each equal to $\frac{1}{2} h_a$ of $\triangle ABC$.

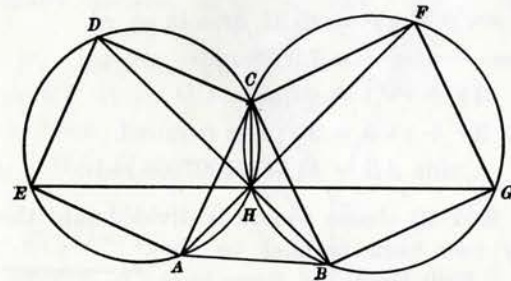
$$\therefore \text{area } \triangle AA'K : \text{area } \triangle ABC = KL : BC = 3 : 4.$$

12. With the medians of a given triangle a new triangle is constructed. The medians of this new triangle are equal to three fourths of the corresponding sides of the given triangle.

Solution. Using diagram in #11, we have $C'L = \frac{1}{3} KL$.

$$\therefore KL = \frac{3}{4} KC' = \frac{3}{4} BC = \frac{3}{4} a.$$

13.* On the side AC of any $\triangle ABC$ the square $ACDE$ is constructed, and on the side BC the square $BCFG$. Prove that the lines AF , BD , and EG are concurrent.



Circumscribe circles about the squares, denoting the second intersection point by H . Join A , B , C , D , E , and F to H .

Since the diagonals EC and CG would be diameters, angles EHC and CHG would be right angles (inscribed in semi-circles), and hence EAG is a straight line.

* From "School Science and Mathematics."

$\angle CHA$ is measured by half of three fourths of a circumference; and $\angle CHF$ is measured by half of one fourth of a circumference. Hence $\angle CHA + CHF = 180^\circ$, or AHF is a straight line.

Similarly, DHB is proved to be a straight line.

$\therefore EG, DB$, and AF are concurrent at H .

14. Suppose the curvilinear space inclosed by three equal circles that just touch each other to be .645 of a square chain. Find the side of an equilateral triangle that will just circumscribe the three circles.

Solution. If r be the radius, then $r\sqrt{3}$ is the altitude of triangle abc and $r^2\sqrt{3}$ is the area of abc . $\frac{1}{2}\pi r^2$ is the area of three sectors in abc .

Then $r^2\sqrt{3} - \frac{1}{2}\pi r^2 = 10.32$, area in sq. rd.

Whence $r = 7.9998$ rods.

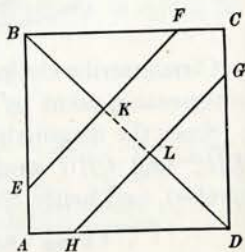
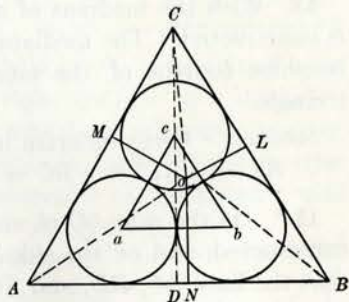
$3r + r\sqrt{3} = \text{altitude, } CD$.

$r\sqrt{3} : 3r + r\sqrt{3} = 2r : \text{side required.}$

$\therefore \text{side } AB = 43.71171997968$ rods.

15. A field 12 chains square is divided into three equal parts by two lines parallel to the diagonal. Find length of these lines and the distance between them.

Solution. The area of field is 12^2 or 144 square chains. Then area of GHD is $\frac{1}{3}$ of 144, or 48 square chains, and since the area of half a square is $\frac{1}{2}$ of the square of the diagonal, we have



$GH = \sqrt{(48 \times 4)}$, or $8\sqrt{3}$ or 13.8564064 ch.,

which is also the measure of EF , and of $BK + LD$.

But $BD = \sqrt{(144 \times 2)}$, or $12\sqrt{2}$ chains;

hence $KL = 12\sqrt{2} - 8\sqrt{3}$, or 3.1141568 chains.

16. There is a garden 100 feet square. On the northeast corner stands a tree 50 ft. high; one on the northwest corner 40 ft. high; one on the southwest corner 20 ft. high; and on the southeast corner one whose height is to be found, and also the length of a pole that will reach from a point within the garden to the tops of the four trees.

Solution. The garden represented by the diagram is viewed from an elevation towards the southwest and is easily understood.

Draw HI perpendicular to KF at its mid-point; then

$KG : FG = HJ : JI$,

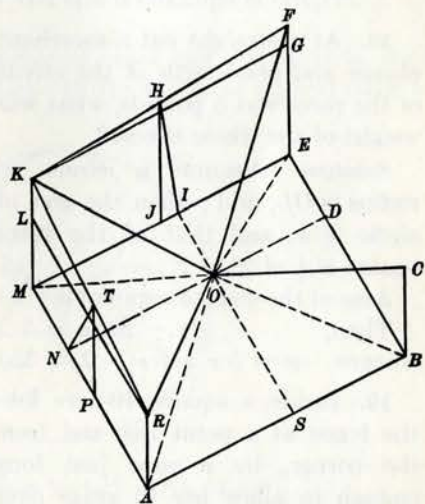
or

$100 : 10 = 45 : JI = 4\frac{1}{2}$.

In a similar way,

$100 : 20 = 30 : NP = 6$.

Draw IS parallel to EB and ND parallel to ME . Then the pole will have its base at O , the point of intersection of the two perpendiculars from I and N , which are equidistant from the tops of the trees, K, F and R, K . J and P are the mid-points of ME and MA .



Hence IE is $45\frac{1}{2}$; MI , $54\frac{1}{2}$; MN , 44; NA , 56; and

$$MO^2 = (54\frac{1}{2})^2 + 44^2 = 4906.25;$$

$$KO = \sqrt{(4906.25 + 40^2)} = 80.66 \text{ feet, length of pole.}$$

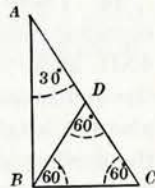
$$OB^2 = (45\frac{1}{2})^2 + 56^2 = 5206.25;$$

$$CB = \sqrt{(80.66^2 - 5206.25)} = 36.06 \text{ feet, height of tree.}$$

17. A ladder 26 feet long leans against a perpendicular wall at an angle of 30° . How far from its mid-point to the base of the wall?

Solution. In a right triangle the side opposite $\angle 30^\circ$ is half the hypotenuse. BC and CD being equal, $\triangle CBD$ and CDB are equal.

$\therefore \triangle CBD$ is equilateral and $BD = 13$ feet.



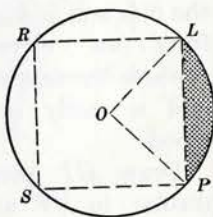
18. At a straight cut a merchant took off a segment of a cheese and one fourth of the circumference. If the weight of the piece was 5 pounds, what was the weight of the whole cheese?

Solution. Assume a circle whose radius is OL , or 1; then the area of the circle is π , and that of the inscribed square is $\frac{1}{2}$ of $2^2 = 2$.

Area of the shaded segment is $\frac{1}{4}(\pi - 2)$.

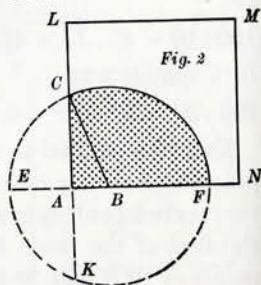
Then, $\frac{1}{4}(\pi - 2) : \pi = 5 : w$,

whence $w = 5\pi \div \frac{1}{4}(\pi - 2) = 55.0387$ lbs.



19. Inside a square 10-acre lot a cow was tethered to the fence at a point one rod from the corner, by a rope just long enough to allow her to graze over an acre of ground; how long was the rope?

Solution. Area of $AFC + AFK = 320$ sq. rd., and adding area of segment, CEK , gives area of circle, and radius to be found.



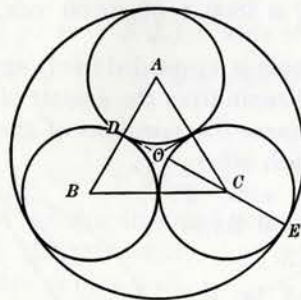
Let $r = BC$, the radius; then $\pi r^2 =$ the area of circle. Obviously, $AC = \sqrt{(r^2 - 1)}$, $EA = r - 1$, and $CK = 2\sqrt{(r^2 - 1)}$.

Substituting these values in formula for segment of a circle, we get area

$\pi r^2 = [(r - 1)^3 \div 4\sqrt{(r^2 - 1)}] + \frac{2}{3}(r - 1)[2\sqrt{(r^2 - 1)}] + 320$. Solving, we find $r = 13.65$ rods.

20. A circular lot 15 rods in diameter is to have three circular grass beds just touching each other and the large boundary; what must be the distance between their centers and how much ground is left in the curvilinear space about the main center?

Solution. As the diameter of outer circle is 15 rods, the radius, OE , is $7\frac{1}{2}$ rods. Let r be the radius of a small circle; then BC is $2r$, and



$$CD = \sqrt{(4r^2 - r^2)}, \text{ or } r\sqrt{3}.$$

$$CO = \frac{2}{3}r\sqrt{3}.$$

Hence $\frac{2}{3}r\sqrt{3} + r = 7\frac{1}{2}$
and $r = 3.4807$.

The distance from center to center is $2r$, or 6.961 rods, and

$$r\sqrt{3} - \frac{1}{2}\pi r^2 = 1.95 \text{ sq. rd., area required.}$$

21. The area of a rectangular field is 30 acres and its diagonal is 100 rods; find length and breadth without algebra.

Solution. In the diagram, let the rectangle FC be the field whose diagonals bisect each other in O .

Area of the right triangle, ABC , or half the field, is 2400, and the altitude

$$CP = 2400 \div 50, \text{ or } 48 \text{ rods.}$$

In the right triangle, OPC , we have OC and CP ; hence

$$OP = \sqrt{(50^2 - 48^2)}, \text{ or } 14,$$

and BP is $50 - 14$, or 36. Then we have

$$BC = \sqrt{(48^2 + 36^2)}, \text{ or } 60 \text{ rods, width of field,}$$

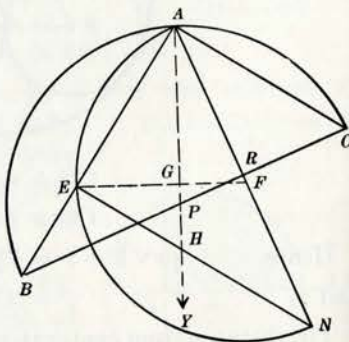
and

$$FB \text{ is } 4800 \div 60, \text{ or } 80 \text{ rods, length of field.}$$

22. A right triangle is suspended freely against a wall from its right angle, and again from the greater of its acute angles, and in these two cases the positions of the hypotenuse are at right angles to each other; find the shorter arm, if the hypotenuse is 60 times the square root of 3.

Solution. Let A be the point of suspension for both positions of the same triangle; AY , the line of gravity which coincides with the medians, AP of the $\triangle BAC$ and AH of the $\triangle AEN$. Hence,

$$AP = PB \text{ and } EH = HN,$$



and $\angle B = \angle BAP$ (AP being equal to PB) $= \angle N$, because all are the same acute angle. The $\triangle AEN$ and AEH , being equiangular, are similar and

$$AE : EH = EN : AE, \\ \text{from which } \overline{AE^2} = EH \times EN; \text{ but } EH = \frac{1}{2} EN. \text{ Therefore,} \\ 2 \overline{AE^2} = \overline{EN^2}.$$

From the properties of the right triangle,

$$\overline{AE^2} + \overline{EN^2} = \overline{AN^2} = (60\sqrt{3})^2 = 3 \times 3600.$$

Substituting the value of $\overline{EN^2}$ above, we get

$$\overline{AE^2} + 2 \overline{AE^2}, \text{ or } 3 \overline{AE^2} = 3 \times 3600, \\ \therefore \overline{AE^2} = 3600, \text{ and } AE \text{ is } 60.$$

23. If from any point within a regular polygon of n sides perpendiculars are drawn to the several sides, the sum of these perpendiculars is equal to n times the apothem.

Solution. Let a = apothem, s = side, and $p_1 + p_2 + \dots + p_n$ the sum of the perpendiculars.

$$\text{Then } \frac{s}{2}(p_1 + p_2 + \dots + p_n) = \text{area of the polygon.}$$

$$\text{Also } \frac{s}{2} \cdot a \cdot n = \text{area of the polygon.}$$

$$\therefore \frac{s}{2}(p_1 + p_2 + \dots + p_n) = \frac{s}{2} na.$$

$$\therefore p_1 + p_2 + \dots + p_n = na.$$

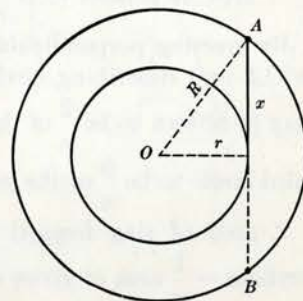
24. The area of the ring included between two concentric circles is equal to the area of a circle whose radius is one half a chord of the outer circle drawn tangent to the inner.

Solution. Let R and r denote the radii of the concentric circles and x one half the chord AB .

$$\text{Area of outer circle} = \pi R^2.$$

$$\text{Area of inner circle} = \pi r^2.$$

$$\text{Area of ring} = \pi(R^2 - r^2).$$



But x being a side of a right triangle, we have

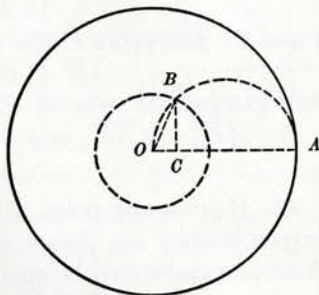
$$x^2 = R^2 - r^2.$$

$$\therefore \text{area of ring} = \pi x^2.$$

25. To divide a circle into n equivalent parts by circumferences concentric with it.

Let O be the center and OA the radius of the given circle to be divided into n parts.

Divide OA into n parts, taking C as the first point of division from O . At C erect a perpendicular meeting the circumference on OA at B . Describe the $\odot$ with radius OB .



The area of this $\odot$ will be $\frac{1}{n}$ the area of the given circle.

$$\overline{OB}^2 = \overline{OC}^2 + \overline{CB}^2 = \overline{OC}^2 + OC \cdot CA$$

$$= OC(OC + CA) = \frac{1}{n} OA \cdot OA = \frac{1}{n} \overline{OA}^2.$$

$$\therefore \pi \overline{OB}^2 = \frac{1}{n} \pi \overline{OA}^2.$$

But $\pi \overline{OB}^2$ is the area of the $\odot$ of which OB is the radius, and $\pi \overline{OA}^2$ is the area of the given $\odot$.

$\therefore$ area of smaller $\odot$ is $\frac{1}{n}$ the area of the given $\odot$.

By erecting perpendiculars at the other points of division in OA and describing circles the area of the second circle may be shown to be $\frac{2}{n}$ of the given circle and the area of the third circle to be $\frac{3}{n}$ of the given circle, etc.

$\therefore$ area of ring formed by the first and second circumferences = $\frac{1}{n}$ area of given circle.

26. If the radius of a circle is r , what is the radius of a concentric circle which divides it into two equivalent parts.

Solution. Let r' be the required radius. From the previous solution we have

$$(r')^2 = \overline{OB}^2 = \frac{1}{n} \overline{OA}^2.$$

Substituting values,

$$(r')^2 = \frac{1}{n} r^2.$$

$$\therefore r' = r \sqrt{\frac{1}{n}} = \frac{r}{\sqrt{n}}.$$

If
then

$$n = 2,$$

$$r' = \frac{r}{\sqrt{2}}.$$

27. If a circle is circumscribed about any triangle, the feet of the perpendiculars dropped from any point in the circumference to the sides of the triangle lie in a straight line.

Let ABC be the $\triangle$, P the point in the circumference, PF , PE , PD the $\perp$ s upon the sides.

Draw EF , ED , BP , and PC .

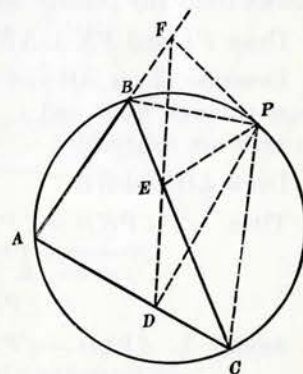
$BFPE$ is concyclic, since two opposite $\angle$ s are rt. $\angle$ s.

$$\therefore \angle FEP = \angle FBP = \angle ACP.$$

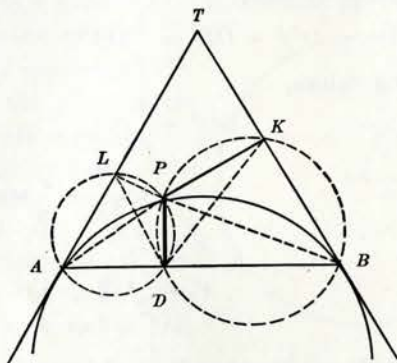
But $\angle ACP$ is the supplement of $\angle DEP$.

$\therefore \angle FEP$ is the supplement of $\angle DEP$.

$\therefore FED$ is a straight line.



28. The perpendicular from any point of a circumference upon a chord is the mean proportional between the perpendiculars from the same point upon the tangents drawn at the extremities of the chord.



Let AB be the chord, AT and BT tangents. $PD \perp AB$ drawn from any point P on the arc AB .

Draw PL and $PK \perp AT$ and BT , respectively.

Describe $\odot$ on BP and AP as diameters. These $\odot$ will pass through K, D and L, D , respectively, since the quadrilaterals are concyclic.

Draw LD and KD .

Then $\angle PKD = \angle PBD = \angle LAP = \angle PDL$.

[Measured by $\frac{1}{2}$ arc PD] [Measured by $\frac{1}{2}$ arc AP] [Measured by $\frac{1}{2}$ arc LP]

$\therefore \angle PKD = \angle PDL$.

Again $\angle PLD = \angle PAD = \angle KBP = \angle PDK$.

[Measured by $\frac{1}{2}$ arc PD] [Measured by $\frac{1}{2}$ arc PB] [Measured by $\frac{1}{2}$ arc PK]

$\therefore \angle PDK = \angle PLD$.

$\therefore \triangle LPD$ and DPK are similar.

$\therefore PL : PD = PD : PK$.

29. If the three perpendiculars drawn from the vertices of the triangle ABC , intersecting the opposite sides in the points A', B', C' , respectively, they are concurrent, then

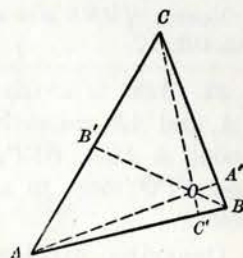
$$\overline{AB'}^2 - \overline{B'C'}^2 + \overline{CA'}^2 - \overline{A'B'}^2 + \overline{BC'}^2 - \overline{C'A'}^2 = 0.$$

$$\text{Proof. } \overline{AB'}^2 - \overline{B'C'}^2 = \overline{AO}^2 - \overline{OC}^2.$$

$$\overline{CA'}^2 - \overline{A'B'}^2 = \overline{OC}^2 - \overline{OB}^2.$$

$$\overline{C'B'}^2 - \overline{AC'}^2 = \overline{BO}^2 - \overline{AO}^2.$$

$$\therefore \overline{AB'}^2 - \overline{B'C'}^2 + \overline{CA'}^2 - \overline{A'B'}^2 + \overline{BC'}^2 - \overline{C'A'}^2 = 0.$$



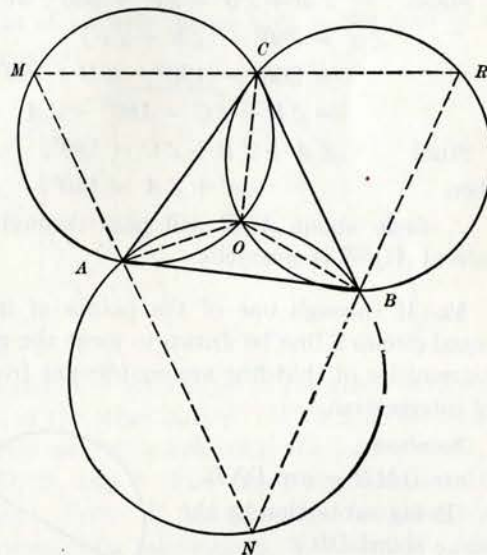
30. Given the triangle ABC and O any point within it. Circumscribe circles about the triangles AOB, BOC, COA . The three circumferences are the loci of the vertices of a triangle of constant form whose sides pass through the points A, B, C .

Proof. Draw line MN through A from circle to circle. Produce MC and NB to meet in R .

$\angle AOB$ and $\angle AOC$ are supplements of $\angle M$ and N , respectively.

$\therefore \angle BOC$ is the supplement of R .

$\therefore R$ lies on circle OBC .

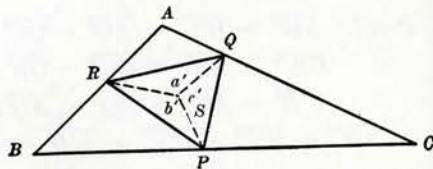


$\therefore \angle M, N$, and R are constant since $\angle$ about O are constant.
 $\therefore \triangle MNR$ is of constant form.

NOTE. $\triangle MNR$ is a maximum when its sides are perpendicular to OA, OB, OC .

31. ABC is a triangle and P, Q, R are points on BC, CA , and AB , respectively. Show that the circles described about $\triangle AQR, BRP$, and CPQ meet in a point.

Describe circles about $\triangle BRP$ and CPQ intersecting in S .



Then $\angle B + \angle b' = 180^\circ, \angle C + \angle c' = 180^\circ$.

Since $\angle a' + \angle b' + \angle c' = 360^\circ$, we have

$$\begin{aligned}\angle a' &= 360^\circ - (\angle b' + \angle c') \\ &= 360^\circ - (180^\circ - \angle B + 180^\circ - \angle C) \\ &= \angle B + \angle C = 180^\circ - \angle A.\end{aligned}$$

Since $\angle A + \angle B + \angle C = 180^\circ$,

then $\angle a' + \angle A = 180^\circ$.

$\therefore$ circle about AQR will pass through S , since quadrilateral $AQSR$ is concyclic.

32. If through one of the points of intersection of two equal circles a line be drawn to meet the circumferences, the extremities of that line are equidistant from the other point of intersection.

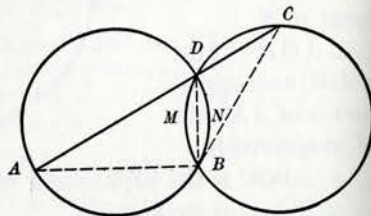
Solution.

are $DMB = \text{arc } DNB$.

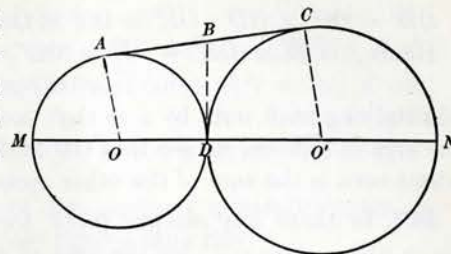
(Being subtended by the same chord DB .)

$\therefore \angle B = \angle C$.

$\therefore AB = BC$.



33. If two circles are tangent externally, the common tangent is a mean proportional between the diameters.



Solution.

$AB = BD = BC$.

BO bisects $\angle ABD$ and BO' bisects $\angle DBC$.

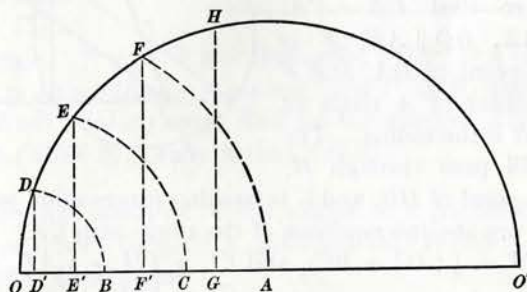
$\therefore \angle OBO' = 90^\circ$.

$\therefore OD : BD = BD : DO'$.

$\therefore 2(OD) : 2(BD) = 2(BD) : 2(DO')$.

$\therefore MD : AC = AC : DN$.

34.* Prove the correctness of the following method for finding the radius of a circle whose area is the sum of the areas of several circles:



Let OA be the radius of the largest of the given circles and OB, OC the radii of the other circles. With A as center and AO as radius draw an arc intersecting the circles at D, E , and F . From D, E , and F draw the perpendiculars DD', EE', FF' , to OA . From F' measure $F'G = OD' + OE'$. At G draw a perpendicular intersecting the arc at H . Then the line OH is the desired radius.

* From "School Science and Mathematics."

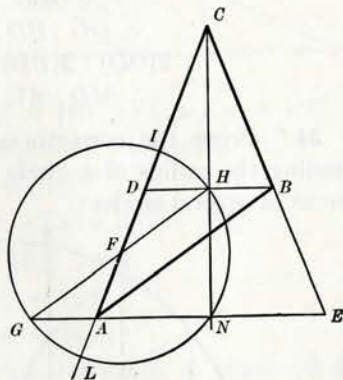
$$\overline{OD}^2 = OO' \times OD'; \overline{OE}^2 = OO' \times OE'; \overline{OF}^2 = OO' \times OF'.$$

Hence, $\overline{OD}^2 + \overline{OE}^2 + \overline{OF}^2 = (OD' + OE' + OF')OO'$
 $= OG \times OO' = \overline{OH}^2.$

Multiplying each term by π so that each term will represent the area of a circle, we see that OH is the radius of the circle whose area is the sum of the other areas.

35.* Is there any simple proof for the formula $S^2 = s(s-a)(s-b)(s-c)$ for the area of a triangle in terms of its sides which does not involve trigonometry nor the complicated algebraic manipulations which are unavoidable when we multiply the base by the altitude as in the text-books?

In the figure, ABC is the given triangle, CB is prolonged so that $CE = CA$, $CN \perp AE$, $BD \parallel AE$, F is the mid-point of DA , and F is the center of a circle of which FN is the radius. The circle will pass through H , the mid-point of DB , and G is its other intersection with AE . I and L are the intersections of the circle with CA .



$$CF = \frac{1}{2}(AC + BC), \text{ and } FL = FH = \frac{1}{2}AB,$$

$$\text{so that } CL = \frac{1}{2}(AC + BC + AB) = s.$$

$$DL = AI = s - BC, CI = s - AB, \text{ and } AL = s - AC.$$

$$CN \times AN = \text{Area } \triangle ACE, HN \times AN = \text{Area } \triangle ABE.$$

Subtracting:

$$AN(CN - HN) = AN \times CH = \text{Area } \triangle ACB. \quad (1)$$

$$CH \times DH = \text{Area } \triangle CDB, HN \times DH = \text{Area } \triangle ADB.$$

Adding,

$$DH(CH + HN) = GA \times CN = \text{Area } \triangle ACB. \quad (2)$$

* From "School Science and Mathematics."

Multiplying equations (1) and (2),

$$GA \times AN \times GH \times CN = (\text{Area } \triangle ACB)^2 = S^2.$$

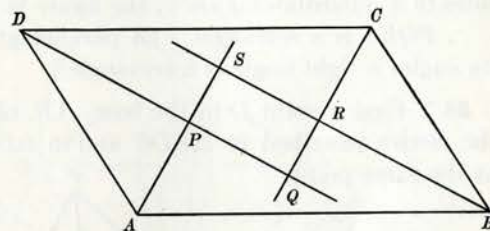
$$\text{But } CN \times CH = GL \times CI \text{ and } GA \times AN = AL \times AI \\ = AL \times DL.$$

$$S^2 = CL \times CI \times AL \times DL = s(s-a)(s-b)(s-c).$$

36.* If the bisectors of the angles of a parallelogram are drawn, how many different figures can arise?

Before trying this exercise be sure the question is understood. The question is not *how many* different triangles, trapezoids, etc.,

can be counted in the figure, but what is the nature of the polygon $PQRS$? Thus, if $ABCD$ is a rhomboid, then $PQRS$



is a rectangle. If $ABCD$ is a rectangle, $PQRS$ is a square. If $ABCD$ is a rhombus or a square, then P, Q, R , and S coincide. What relation must hold for the sides of $ABCD$ in order that $PQRS$ lie wholly within $ABCD$?

37.* The four bisectors of the angles of a parallelogram, $ABCD$, are drawn forming a quadrilateral, $PQRS$. Prove that $PQRS$ is a rectangle.

Solution. See diagram to previous exercise.

Hyp. $ABCD$ is a parallelogram. AS bisects $\angle BAD$. BS bisects $\angle ABC$. CQ bisects $\angle BCD$. DQ bisects $\angle CDA$.

Con. $PQRS$ is a rectangle.

Proof. $\angle CDA + \angle DAB = 180^\circ$. (Two consecutive angles of a parallelogram are supplementary.)

* From "School Science and Mathematics."

$\angle ADQ = \frac{1}{2} \angle CDA$. (DQ bisects $\angle CDA$. — *Hyp.*)

$\angle DAP = \frac{1}{2} \angle DAB$. (AS bisects $\angle BAD$. — *Hyp.*)

$\therefore \angle ADQ + \angle DAP = 90^\circ$. (Halves of supplementary angles are complementary.)

$\angle APD = 90^\circ$. (The sum of the $\angle$ s of a $\triangle$ is 180° .)

$\angle SPQ = 90^\circ$. (Vertical $\angle$ s are equal.)

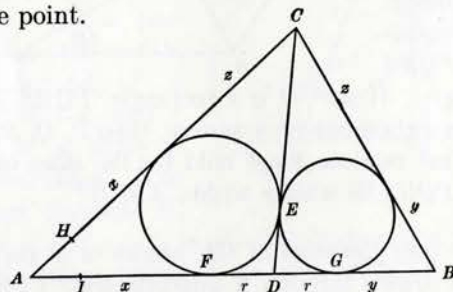
(In a similar way we may prove that $\angle SRQ = 90^\circ$, $\angle ASB = 90^\circ$, $\angle DQC = 90^\circ$.)

$DQ \parallel SB$. (Lines $\perp$ to the same line, QC , are $\parallel$.)

$PQRS$ is a parallelogram. (*Definition*: If the opposite sides of a quadrilateral are $\parallel$, the figure is a parallelogram.)

$\therefore PQRS$ is a rectangle. (A parallelogram having one of its angles a right angle is a rectangle.)

38.* Find a point D in the base, AB , of $\triangle ABC$ such that the circles inscribed in $\triangle ADC$ and in $\triangle DBC$ will touch CD at the same point.



Solution. In the figure, if CD is the required line, then both circles are tangent to CD at E and $DE = DF = DG = r$. Let s = the semi-perimeter of the given $\triangle ABC$; $AF = x$; $BG = y$. Then

$$x + y + z + r = s \quad \text{and} \quad x + z = b$$

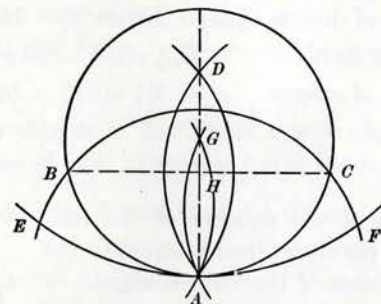
Hence $BD = y + r = s - b$ and from this equation we can determine the point D ; in fact, D is the point of tangency of the circle inscribed in $\triangle ABC$.

* From "School Science Mathematics."

Also $s - b$, or BD can be found easily as follows: With C as a center and CB as a radius make $CH = CB$; with A as a center and AH as a radius make $AI = AH$; then D is the mid-point of BI .

39.* Find the center of a circle using compasses only.

Solution. With A , any point on the circle, as a center and with any convenient radius draw a circle cutting the given circle in points B and C . With B and C as centers and AB as a radius draw two arcs intersecting at D . With D as



center and AD as radius draw an arc intersecting at E and F , the circle whose center is A . With E and F as centers and EA as radius draw arcs intersecting at G . This is the center of the given circle.

Proof. $\triangle DEA \sim \triangle GEA$ because both are isosceles and $\angle EAG$ is common. Then $\overline{EA}^2 = AD \times AG$. Call H the intersection point of AD and BC . Then $AD = 2AH$; hence $\overline{EA}^2 = 2AH \times AG$, or

$$\begin{aligned} 2AH \times AG &= \overline{EA}^2 = \overline{BA}^2 = \overline{AH} + \overline{HB}^2 = \overline{AH}^2 + \overline{BG} - \overline{HG}^2 \\ &= \overline{AH}^2 + \overline{BG}^2 - (\overline{AG} - \overline{AH})^2 \\ &= \overline{BG}^2 - \overline{AG}^2 + 2AH \times AG. \end{aligned}$$

Hence $BG = AG$. Similarly, $CG = AG$.

* From "School Science Mathematics."

SOLID GEOMETRY

1. A square $ABCD$ and its circumscribing circle revolve about the diagonal AC as an axis. Compare the volumes and surfaces of the solids generated, the diagonal AC being 6 feet.

Solution. The square generates a double cone, altitude 3 ft. and diameter of base 6 ft., while the circle generates a sphere.

$$\text{Vol. of double cone} = 2(9\pi) = 18\pi.$$

$$\text{Surface of double cone} = 2(9\pi\sqrt{2}) = 18\pi\sqrt{2}.$$

$$\text{Vol. of sphere} = \frac{1}{6}\pi(6)^3 = 36\pi.$$

$$\text{Surface of sphere} = 4[\frac{1}{4}\pi(6)^2] = 36\pi.$$

$\therefore$ vol. of sphere = 2 times vol. of double cone and surface of sphere = $\sqrt{2}$ times surface of double cone.

2. The sum of the squares of the four diagonals of a parallelepiped is equal to the sum of the squares of the twelve edges.

Solution. Since $ABGH$ is a parallelogram,

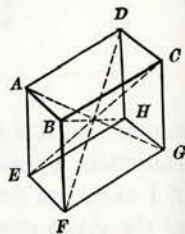
$$\overline{AG}^2 + \overline{BH}^2 = \overline{AB}^2 + \overline{BG}^2 + \overline{GH}^2 + \overline{HA}^2.$$

$$\overline{EC}^2 + \overline{FD}^2 = \overline{EF}^2 + \overline{FC}^2 + \overline{DC}^2 + \overline{DE}^2.$$

$$\therefore \overline{AG}^2 + \overline{BH}^2 + \overline{EC}^2 + \overline{FD}^2 = \overline{AB}^2 + \overline{EF}^2 + (\overline{BG}^2 + \overline{FC}^2) + \overline{GH}^2 + \overline{DC}^2 + (\overline{HA}^2 + \overline{DE}^2).$$

$$\therefore \overline{AG}^2 + \overline{BH}^2 + \overline{EC}^2 + \overline{FD}^2 = \overline{AB}^2 + \overline{EF}^2 + \overline{BF}^2 + \overline{FG}^2 + \overline{GC}^2 + \overline{CB}^2 + \overline{GH}^2 + \overline{DC}^2 + \overline{AD}^2 + \overline{DH}^2 + \overline{HE}^2 + \overline{EA}^2.$$

3. If the angles at the vertex of a triangular pyramid are right angles, and the lateral edges are equal, prove that the sum of the perpendiculars on the lateral faces from any point in the base is constant.



Solution. Let $O-ABC$ be the triangular pyramid. Through P draw in the plane of the base parallels to AB , AC , BC .

$$\text{Then } \frac{PK}{CO} = \frac{PD}{CD} = \frac{rA}{CA}.$$

[Similar $\triangle$.]

$$\text{Also } \frac{PM}{AO} = \frac{xC}{AC}$$

$$\text{and } \frac{PL}{BO} = \frac{uC}{BC}.$$

Adding and noting that $BO = AO = CO$ and $BC = AC$, we have

$$\frac{PK + PM + PL}{CO} = \frac{rA + xC + uC}{AC}. \quad (a)$$

Since $uC = Px$ (being opposite sides of a $\square$) and $Px = xr$ (sides of an equilateral $\triangle$ similar to $\triangle CBA$),

$$\therefore uC = xr$$

Substituting values in (a), we have

$$\frac{PK + PM + PL}{CO} = \frac{rA + xC + xr}{AC} = \frac{AC}{AC} = 1.$$

$$\therefore PK + PM + PL = CO, \text{ a constant.}$$

4. In the trihedral angle $X-ABC$, XD is the bisector of the face angle BXC .

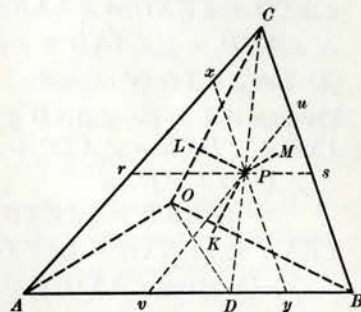
Prove: $\angle AXD \leq \frac{1}{2}(\angle AXB + \angle AXC)$ according as $\angle AXD$ is acute, right, or obtuse.

(1) Let $\angle AXD$ be acute.

In the plane AXD draw XE making $\angle DXE = \angle AXD$.

Since $\angle BXD = \angle DXC$, $\angle AXD = \angle DXE$ and dihedral $\angle B-XD-E = \text{dihedral } \angle C-XD-A$, $\angle BXE = \angle CXA$.
 $\angle AXE < \angle AXB + \angle EXB$.

$$\therefore \angle AXD < \frac{1}{2}(\angle AXB + \angle AXC).$$



(2) Let $\angle AXD$ be a right angle.

Produce AX to E . $\angle BXE = \angle CXA$ as above.

$$\angle AXD + \angle EXD = \angle AXB + \angle EXB.$$

$$\therefore \angle AXD = \frac{1}{2}(\angle AXB + \angle AXC).$$

(3) Let $\angle AXD$ be obtuse.

Produce AX to E . $\angle AXD + \angle DXE = \angle AXB + \angle EXB$.

$$\angle AXD + \angle DXE = \angle AXC + \angle EXC.$$

$$\therefore \angle AXD + \angle DXE$$

$$= \frac{1}{2}\angle AXB + \frac{1}{2}\angle EXB + \frac{1}{2}\angle AXC + \frac{1}{2}\angle EXC. \quad (a)$$

$$\angle EXD < \frac{1}{2}\angle EXB + \frac{1}{2}\angle EXC \text{ [by (1)]}. \quad (b)$$

$$\therefore (a)-(b) \text{ gives } \angle AXD > \frac{1}{2}(\angle AXB + \angle AXC).$$

5. In a sphere is inscribed a cone whose volume is one fourth the volume of the sphere. Find the altitude and the radius of the base of the cone.

Let R = radius of sphere, h = altitude of cone, and r = radius of its base.

$$\text{Then } r = [R^2 - (h - R)^2]^{\frac{1}{2}} = [2hR - R^2]^{\frac{1}{2}}.$$

$$\text{Also } \frac{1}{3}\pi hr^2 = \frac{1}{4}(\frac{4}{3}\pi R^3) \text{ or } hr^2 = R^3.$$

$$\therefore h(2hR - h^2) = R^3.$$

$$\therefore h^3 - 2h^2R + R^3 = (h - R)(h^2 - hR - R^2) = 0.$$

$$\therefore h = R \text{ or } \frac{R}{2}(\sqrt{5} + 1).$$

$$\text{Then } r = R, \text{ or } \frac{R}{2}[2\sqrt{5} - 2]^{\frac{1}{2}}.$$

6. Find the weight of a sphere of radius R , which floats in a liquid of specific gravity s , with one fourth of its surface above the surface of the liquid.

Solution. Let Z = area of zone = $2\pi RH$.

$$\frac{1}{4}S = \pi R^2 = 2\pi RH.$$

$$\therefore H = \frac{1}{2}R.$$

$$\text{Volume out of liquid} = \pi H^2 \left(R - \frac{H}{3} \right).$$

$$= \frac{\pi R^2}{4} \left(R - \frac{R}{6} \right).$$

$$= \frac{\pi R^2}{4} \cdot \frac{5R}{6} = \frac{5\pi R^3}{24}.$$

$$\text{Volume of liquid displaced} = \frac{4}{3}\pi R^3 - \frac{5}{24}\pi R^3 = \frac{9}{8}\pi R^3.$$

$$\therefore \text{weight of sphere} = \frac{9}{8}\pi R^3 s.$$

7. Given the edge of a regular tetrahedron equal to a , to find the radius of the sphere that touches the six edges.

The diameter is the altitude of an isosceles triangle having for equal sides the altitudes of two of the faces, i.e., equal sides equal to $\frac{a}{2}\sqrt{3}$.

The base of this isosceles triangle is a .

$$\text{Hence radius} = \frac{1}{2} \text{ diameter} = \frac{1}{2} \cdot \frac{a}{2}\sqrt{2} = \frac{a}{4}\sqrt{2}.$$

8.* A point P is within a regular tetrahedron and its distances from the vertices are 3, 4, 5, 6 ft. Find the volume of the tetrahedron.

Solution. Let $O-ABC$ be the tetrahedron with side $2e$. P is a point such that $OP = 3$, $AP = 5$, $BP = 6$, $CP = 4$. Using the customary rectangular axes, we say the coördinates

of O are $(0, 0, 0)$, of A are $(e, e\sqrt{3}, 0)$, of B are $(e, \frac{e}{\sqrt{3}}, \frac{2e\sqrt{2}}{3})$,

of C are $(2e, 0, 0)$, and of P are (k, l, m) . Then

$$(1) \quad k^2 + l^2 + m^2 = 9$$

$$(2) \quad (k - 2e)^2 + l^2 + m^2 = 16$$

$$(3) \quad (k - e)^2 + (l - e\sqrt{3})^2 + m^2 = 25$$

$$(4) \quad (k - e)^2 + \left(l - \frac{e}{\sqrt{3}} \right)^2 + (m - 2e\sqrt{\frac{2}{3}})^2 = 36$$

* From "School Science and Mathematics."

This set of equations can be simplified by subtracting each equation from the first. Then,

$$k = \frac{(4e^2 - 7)}{4e}; l = \frac{(4e^2 - 25)}{4e\sqrt{3}};$$

$$m = \frac{(2e^2 - 29)}{2e\sqrt{6}}.$$

If these values are substituted in equation (1), we get

$$12e^4 - 172e^2 + 409 = 0,$$

so that $e^2 = 11.3233$ (or 3.01 if P is outside the tetrahedron). Then $e = 3.365$; $2e = 6.730$; and volume = 35.923.

9. The straight lines joining the middle points of the opposite edges of a tetrahedron meet in a point, and are each bisected by the point.

Solution. Let $O-ABC$ be the given tetrahedron. Take O' the point of the intersection of the medians of the $\triangle ABC$. Draw OO' .

Since LM and OO' are not parallel and in the same plane, they will meet in some point S . Draw $MK \parallel OO'$. Since MK bisects OC , it also bisects $O'C$.

$$\therefore O'K = \frac{1}{2} O'C = LO'.$$

Since $\triangle LO'S$ and LKM are similar,

$$\therefore LS = SM.$$

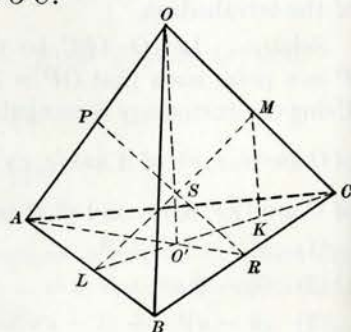
Also since $SO' = \frac{1}{2} MK$, and $MK = \frac{1}{2} OO'$,

$\therefore SO' = \frac{1}{4} OO'$, or LM is bisected by OO' at point S .

Likewise we may show that PR is bisected by OO' in point S .

$\therefore PR$ and LM bisect each other in point S .

Likewise the lines joining the middle points of the other pairs of opposite edges are each bisected at S .



TRIGONOMETRY

1. If A, B, C are the angles of a $\triangle$, prove $\sin A + \sin B + \sin C = 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C$.

Solution. Since $A + B + C = 180^\circ$, we have

$$\sin(A + B) = \sin C, \text{ and } \sin \frac{1}{2}(A + B) = \cos \frac{1}{2} C$$

$$\begin{aligned} \text{Now } \sin A + \sin B &= 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B) \\ &= 2 \cos \frac{1}{2} C \cos \frac{1}{2}(A - B) \end{aligned}$$

$$\begin{aligned} \text{and } \sin C &= 2 \sin \frac{1}{2} C \cos \frac{1}{2} C \\ &= 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2} C. \end{aligned}$$

$$\begin{aligned} \therefore \sin A + \sin B + \sin C &= 2 \cos \frac{1}{2} C \cos \frac{1}{2}(A - B) + 2 \cos \frac{1}{2} C \cos \frac{1}{2}(A + B) \\ &= 2 \cos \frac{1}{2} C [\cos \frac{1}{2}(A - B) + \cos \frac{1}{2}(A + B)] \\ &= 2 \cos \frac{1}{2} C [2 \cos \frac{1}{2} A \cos \frac{1}{2} B] \\ &= 4 \cos \frac{1}{2} A \cos \frac{1}{2} B \cos \frac{1}{2} C. \end{aligned}$$

2. If A, B, C are the angles of a $\triangle$, prove $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C$.

Solution. Since $A + B + C = 180^\circ$, we have

$$\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$$

$$\text{and } \cos C = 1 - 2 \sin^2 \frac{1}{2} C.$$

$$\begin{aligned} \therefore \cos A + \cos B + \cos C &= 1 + 2 \sin \frac{1}{2} C [\cos \frac{1}{2}(A - B) - \sin \frac{1}{2} C] \\ &= 1 + 2 \sin \frac{1}{2} C [\cos \frac{1}{2}(A - B) - \cos \frac{1}{2}(A + B)] \\ &= 1 + 4 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C. \end{aligned}$$

3. If A, B, C are the angles of a $\triangle$, prove $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Solution. Since $A + B + C = 180^\circ$, we have

$$\begin{aligned} \tan(A + B) &= -\tan C \\ &= \frac{\tan A + \tan B}{1 - \tan A \tan B} \end{aligned}$$

$$\therefore \tan A + \tan B = -\tan C (1 - \tan A \tan B).$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

4. If A, B, C are the angles of a Δ , prove

$$\cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C = \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C.$$

Solution. Since $\frac{1}{2} A + \frac{1}{2} B + \frac{1}{2} C = 90^\circ$, or

$$\frac{1}{2} C = 90^\circ - \frac{1}{2} (A + B),$$

then

$$\cot \frac{1}{2} C = \tan \frac{1}{2} (A + B),$$

$$\cot \frac{1}{2} B = \tan \frac{1}{2} (A + C),$$

$$\cot \frac{1}{2} A = \tan \frac{1}{2} (B + C).$$

$$\therefore \cot \frac{1}{2} A + \cot \frac{1}{2} B + \cot \frac{1}{2} C$$

$$= \tan \frac{1}{2} (A + B) + \tan \frac{1}{2} (A + C) + \tan \frac{1}{2} (B + C).$$

$$\text{By Ex. 3.} = \tan \frac{1}{2} (A + B) \tan \frac{1}{2} (A + C) \tan \frac{1}{2} (B + C)$$

$$= \cot \frac{1}{2} A \cot \frac{1}{2} B \cot \frac{1}{2} C.$$

5. If A, B, C are the angles of a Δ , prove

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C.$$

Solution. Since $C = 180^\circ - (A + B)$,

$$\sin C = \sin (A + B) \text{ and } \cos C$$

$$= -\cos (A + B).$$

$$\text{Then } \sin 2A + \sin 2B = 2 \sin (A + B) \cos (A - B)$$

$$= 2 \sin C \cos (A - B).$$

Also

$$\sin 2C = 2 \sin C \cos C$$

$$= -2 \sin C \cos (A + B).$$

$$\therefore \sin 2A + \sin 2B + \sin 2C$$

$$= 2 \sin C [\cos (A - B) - \cos (A + B)]$$

$$= 2 \sin C [2 \sin A \sin B]$$

$$= 4 \sin A \sin B \sin C.$$

6. Express $\sin 3x$ in terms of $\sin x$.

Solution. $\sin 3x = \sin (2x + x)$

$$= \sin 2x \cos x + \cos 2x \sin x$$

$$= 2 \sin x \cos^2 x + (1 - 2 \sin^2 x) \sin x$$

$$= 2 \sin x (1 - \sin^2 x) + \sin x - 2 \sin^3 x$$

$$= 3 \sin x - 4 \sin^3 x.$$

7. Express $\cos 3x$ in terms of $\cos x$.

Solution. $\cos 3x = \cos (2x + x)$

$$= \cos 2x \cos x - \sin 2x \sin x$$

$$= (2 \cos^2 x - 1) \cos x - 2 \sin^2 x \cos x.$$

$$= 4 \cos^3 x - 3 \cos x.$$

8. Express $\tan 3x$ in terms of $\tan x$.

Solution.

$$\tan 3x = \tan (2x + x)$$

$$= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

$$= \frac{\frac{2 \tan x}{1 - \tan^2 x} + \tan x}{1 - \frac{2 \tan^2 x}{1 - \tan^2 x}}$$

$$= \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}.$$

9. Prove that $\frac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan \frac{A}{2}$.

Solution. Using the relations $\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2}$,

$$1 + \cos A = 2 \cos^2 \frac{A}{2},$$

$$1 - \cos A = 2 \sin^2 \frac{A}{2},$$

we have

$$\frac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \frac{2 \sin^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2}}{2 \cos^2 \frac{A}{2} + 2 \sin \frac{A}{2} \cos \frac{A}{2}}$$

$$= \frac{2 \sin \frac{A}{2} \left(\sin \frac{A}{2} + \cos \frac{A}{2} \right)}{2 \cos \frac{A}{2} \left(\sin \frac{A}{2} + \cos \frac{A}{2} \right)} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \tan \frac{A}{2}.$$

10. A person standing south of a tower, on the same horizontal plane, observes its angle of elevation to be $54^\circ 16'$; he goes east 100 yards, and then finds its angle of elevation is $50^\circ 8'$. Find the height of the tower.

Solution. Let h = height of tower and C and B the first and second positions of the observer.

Then $b = h \cot x$,
 $c = h \cot y$.

Since $c^2 - b^2 = a^2$,
we have

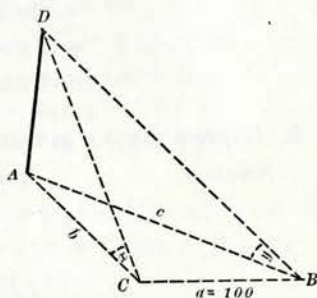
$$\begin{aligned} h &= \frac{a}{\sqrt{\cot^2 y - \cot^2 x}} = \frac{a}{\sqrt{\frac{\cos^2 y}{\sin^2 y} - \frac{\cos^2 x}{\sin^2 x}}} \\ &= \frac{a \sin x \sin y}{\sqrt{\sin^2 x \cos^2 y - \cos^2 x \sin^2 y}} \\ &= \frac{a \sin x \sin y}{\sqrt{(\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y)}} \\ &= \frac{a \sin x \sin y}{\sqrt{\sin(x+y) \sin(x-y)}} \end{aligned}$$

Since

$$\begin{aligned} x &= 54^\circ 16' \text{ and } y = 50^\circ 8', \\ \sin(x+y) &= \sin 104^\circ 24', \\ \sin(x-y) &= \sin 4^\circ 8'. \end{aligned}$$

Solving for h by use of logarithms, we have

$$\begin{aligned} \log a &= 2.00000 \\ \log \sin x &= 9.90942 - 10 \\ \log \sin y &= 9.88510 - 10 \\ \text{colog } \sqrt{\sin(x+y)} &= 0.00693 \\ \text{colog } \sqrt{\sin(x-y)} &= 0.57110 \\ \log h &= 2.37255 \\ \therefore h &= 235.81 \text{ yd.} \end{aligned}$$



11. Solve $\sin^2 x \cos^2 x - \cos^2 x - \sin^2 x + 1 = 0$.

Solution.

$$\sin^2 x \cos^2 x - \cos^2 x - \sin^2 x + 1 = 0.$$

$$\sin^2 x \cos^2 x - (\cos^2 x + \sin^2 x) + 1 = 0.$$

$$\sin^2 x \cos^2 x = 0.$$

$$\sin x \cos x = 0.$$

$$2 \sin x \cos x = 0.$$

$$\sin 2x = 0.$$

$$\therefore 2x = 0^\circ, 180^\circ, 360^\circ, \text{ or } 540^\circ.$$

$$\therefore x = 0^\circ, 90^\circ, 180^\circ, \text{ or } 270^\circ.$$

12. Prove $\sin(A+B+C) = \sin A \cos B \cos C + \sin B \cos A \cos C + \sin C \cos A \cos B - \sin A \sin B \sin C$.

Solution. Since $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\cos(A+B) = \cos A \cos B - \sin A \sin B$, we may substitute $(B+C)$ for B in them as follows:

$$\sin(A+B+C) = \sin A \cos(B+C) + \cos A \sin(B+C).$$

Substituting the values of $\cos(B+C)$ and $\sin(B+C)$, we have

$$\begin{aligned} \sin(A+B+C) &= \sin A [\cos B \cos C - \sin B \sin C] \\ &\quad + \cos A [\sin B \cos C + \cos B \sin C]. \end{aligned}$$

$$\therefore \sin(A+B+C) = \sin A \cos B \cos C - \sin A \sin B \sin C + \sin B \cos A \cos C + \sin C \cos A \cos B.$$

13. Prove $\cos(A+B+C) = \cos A \cos B \cos C - \cos A \sin B \sin C - \cos B \sin A \sin C - \cos C \sin A \sin B$.
Proof similar to that of Ex. 12.

14. Prove $\tan(A+B+C)$

$$= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan B \tan C - \tan A \tan B - \tan A \tan C}.$$

Proof similar to that of Ex. 12 and Ex. 13.

$$15.* \text{ Show that } \sin 3^\circ = \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) \\ + \frac{1}{8}(\sqrt{5} + \sqrt{5} - \sqrt{15 + 3\sqrt{5}}).$$

Solution. Since $3^\circ = 12^\circ - 9^\circ$, $12^\circ = 30^\circ - 18^\circ$, and, for $\theta = 18^\circ$, $2\theta = 90^\circ - 3\theta$, the sine and the cosine of 9° , 12° , 18° , and thus $\sin 3^\circ$ may be found as follows:

$$\text{We have } \sin 2\theta = 2 \sin \theta \cos \theta = \sin(90^\circ - 3\theta) \\ = \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta,$$

$$\text{and } 4 \sin^2 \theta + 2 \sin \theta = 1,$$

$$\sin 18^\circ = \frac{1}{4}(\sqrt{5} - 1), \cos 18^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}.$$

Also from the identities, $\sin \frac{1}{2}A \pm \cos \frac{1}{2}A = \pm \sqrt{1 \pm \sin A}$ for $A = 18^\circ$, we have

$$\sin 9^\circ = \frac{1}{4}(\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}),$$

$$\cos 9^\circ = \frac{1}{4}(\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}).$$

Also

$$\sin 12^\circ = \sin(30^\circ - 18^\circ) = \sin 30^\circ \cos 18^\circ - \cos 30^\circ \sin 18^\circ \\ = \frac{1}{8}(\sqrt{10 + 2\sqrt{5}} - \sqrt{15} + \sqrt{3}),$$

$$\cos 12^\circ = \frac{1}{8}(\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1).$$

Therefore,

$$\begin{aligned} \sin 3^\circ &= \sin(12^\circ - 9^\circ) = \sin 12^\circ \cos 9^\circ - \cos 12^\circ \sin 9^\circ \\ &= \left(\frac{\sqrt{10 + 2\sqrt{5}} - \sqrt{15} + \sqrt{3}}{8} \right) \left(\frac{\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}}{4} \right) \\ &\quad - \left(\frac{\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1}{8} \right) \left(\frac{\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}}{4} \right) \\ &= \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \frac{1}{16}(\sqrt{10} + 4\sqrt{5} \\ &\quad + \sqrt{10} - 4\sqrt{5} - \sqrt{30 + 12\sqrt{5}} \\ &\quad - \sqrt{30 - 12\sqrt{5}}) \\ &= \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) \\ &\quad + \frac{1}{16}(1 - \sqrt{3})(\sqrt{10 + 4\sqrt{5}} + \sqrt{10 - 4\sqrt{5}}). \end{aligned}$$

* From *American Mathematical Monthly*.

By simplifying the expression in the last parenthesis and then collecting under the radical sign, we have, after a simple reduction,

$$\sin 3^\circ = \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) \\ + \frac{1}{8}(\sqrt{5} + \sqrt{5} - \sqrt{15 + 3\sqrt{5}}).$$

16. In the triangle ABC , if $\angle A$ is twice $\angle B$, prove $a^2 = b^2 + bc$.

Solution.

$$\frac{a}{b} = \frac{\sin A}{\sin B} = 2 \sin \frac{1}{2}A \frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A} = 2 \cos \frac{1}{2}A.$$

$$\therefore \cos \frac{1}{2}A = \frac{a}{2b}.$$

$$\text{Also } b^2 = a^2 + c^2 - 2ac \cos B.$$

$$\therefore \cos B = \frac{(a^2 + c^2 - b^2)}{2ac} = \cos \frac{1}{2}A.$$

$$\therefore \frac{a}{2b} = \frac{(a^2 + c^2 - b^2)}{2ac}.$$

Clearing of fractions,

$$a^2(b - c) = b(b^2 - c^2)$$

or

$$a^2 = b(b + c).$$

17. Prove that $\cos 20^\circ \cos 40^\circ \cos 80^\circ = \frac{1}{8}$.

Solution I.

$$\text{Let } \cos 20^\circ \cos 40^\circ \cos 80^\circ = k.$$

Using the formula $2 \cos \alpha \cos \beta = \cos(\alpha + \beta) + \cos(\alpha - \beta)$ this can be written:

$$\frac{1}{2} \cos 40^\circ (\cos 100^\circ + \cos 60^\circ) = k \text{ or,}$$

$$\text{since } \cos 60^\circ = \frac{1}{2}: \quad \frac{1}{4}(2 \cos 100^\circ \cos 40^\circ + \cos 40^\circ) = k.$$

Using the same formula again, we have:

$$\frac{1}{4}(\cos 140^\circ + \cos 60^\circ + \cos 40^\circ) = k.$$

Using the formula

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta),$$

$$\text{then } \frac{1}{4}(2 \cos 50^\circ \cos 90^\circ + \cos 60^\circ) = k.$$

$$\therefore \cos 20^\circ \cos 40^\circ \cos 80^\circ = k = \frac{1}{8},$$

since

$$\cos 90^\circ = 0, \text{ and } \cos 60^\circ = \frac{1}{2}.$$

Solution II.

$$\begin{aligned}\sin 20^\circ &= \sin 160^\circ \\ &= 2 \sin 80^\circ \cos 80^\circ \\ &= 2 (2 \sin 40^\circ \cos 40^\circ) \cos 80^\circ \\ &= 4 (2 \sin 20^\circ \cos 20^\circ) \cos 40^\circ \cos 80^\circ \\ \therefore 1 &= 8 \cos 20^\circ \cos 40^\circ \cos 80^\circ.\end{aligned}$$

18. A quadrilateral $ABCD$ is such a one that a circle can be inscribed in it and another circle circumscribed about it. Prove that

$$\frac{\tan^2 A}{2} = \frac{bc}{ad}.$$

Solution. By the law of cosines:

$$BD^2 = a^2 + d^2 - 2ad \cos A = b^2 + c^2 - 2bc \cos C$$

Since the quadrilateral can be inscribed in a circle,

$$\cos C = -\cos A.$$

$$\text{Hence} \quad \cos A = \frac{(a^2 + d^2 - b^2 - c^2)}{2(ad + bc)}.$$

$$\text{But} \quad \frac{\tan^2 A}{2} = \frac{(1 - \cos A)}{(1 + \cos A)}.$$

Substituting the above value of $\cos A$, and making use of the relation $a + c = b + d$ since a circle may be inscribed within the quadrilateral, we get: $\frac{\tan^2 A}{2} = \frac{bc}{ad}$.

19. Eliminate θ from the equations

$$x = \cos 2\theta + 2 \cos \theta,$$

$$y = \sin 2\theta - 2 \sin \theta.$$

Solution. Given $x = \cos 2\theta + 2 \cos \theta$, (1)

$$y = \sin 2\theta - 2 \sin \theta. \quad (2)$$

From (1) we derive

$$2 \cos^2 \theta + 2 \cos \theta = 1 + x.$$

$$\text{Hence} \quad \cos \theta = \frac{\pm \sqrt{3 + 2x - 1}}{2};$$

Squaring (1) and (2), and adding, we get

$$x^2 + y^2 = 5 + 4 \cos 3\theta.$$

$$\text{Hence,} \quad \cos 3\theta = \frac{x^2 + y^2 - 5}{4}.$$

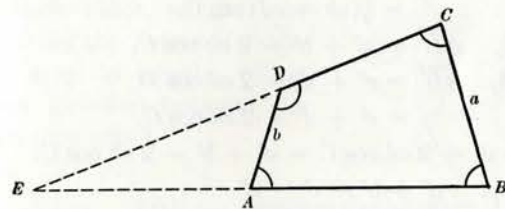
$$\text{But} \quad \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.$$

Therefore,

$$\frac{x^2 + y^2 - 5}{4} = 4 \left(\frac{\pm \sqrt{3 + 2x - 1}}{2} \right)^3 - 3 \left(\frac{\pm \sqrt{3 + 2x - 1}}{2} \right)$$

or, after simplifying, $x^2 + y^2 + 12x + 9 = \pm 2(3 + 2x)^{\frac{3}{2}}.$

20. Find the area of a quadrilateral, given the angles A , B , C , and D and two opposite sides a and b .



Solution. Let $a = BC$ and $b = AD$. $E = 180^\circ - (B + C)$

The $\angle$ at A being supplementary, their sines are equal. The same is true of the $\angle$ at D .

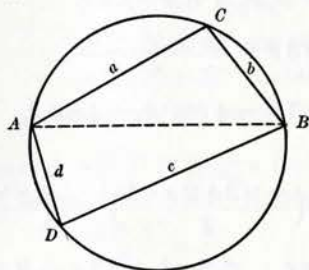
Area of $ABCD$ = area of BCE - area of ADE .

$$\text{Area of } BCE = \frac{a^2 \sin B \sin C}{2 \sin E};$$

$$\text{Area of } ADE = \frac{b^2 \sin A \sin D}{2 \sin E}.$$

$$\therefore \text{area of } ABCD = \frac{a^2 \sin B \sin C - b^2 \sin A \sin D}{2 \sin E}.$$

21. Find the area of an inscribed quadrilateral, given the sides a, b, c , and d .



Solution.

$$\text{Area of } ACBD = \text{area } \triangle ACB + \text{area } \triangle ADB$$

$$= \frac{1}{2} ab \sin C + \frac{1}{2} cd \sin C$$

$$= \frac{1}{2} ab \sin C + \frac{1}{2} cd \sin C$$

$$= \frac{1}{2} (ab + cd) \sin C.$$

$$\text{In } \triangle ABC, \quad \overline{AB}^2 = a^2 + b^2 - 2ab \cos C.$$

$$\text{In } \triangle ADB, \quad \overline{AB}^2 = c^2 + d^2 - 2cd \cos D$$

$$= c^2 + d^2 + 2cd \cos C.$$

$$\therefore c^2 + d^2 + 2cd \cos C = a^2 + b^2 - 2ab \cos C.$$

$$\therefore \cos C = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

$$\sin C = \sqrt{1 - \cos^2 C} = \sqrt{1 - \left[\frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)} \right]^2}$$

$$= \frac{\sqrt{4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2}}{2(ab + cd)}$$

$$= \frac{\sqrt{(2ab + 2cd + a^2 + b^2 - c^2 - d^2)(2ab + 2cd - a^2 - b^2 + c^2 + d^2)}}{2(ab + cd)}$$

$$= \frac{\sqrt{[(a + b)^2 - (c - d)^2][(c + d)^2 - (a - b)^2]}}{2(ab + cd)}$$

$$= \frac{\sqrt{(a + b + c - d)(a + b + d - c)(a + c + d - b)(b + c + d - a)}}{2(ab + cd)}.$$

Let $s = \frac{1}{2}(a + b + c + d)$. Then

$$a + b + c - d = 2(s - d), a + b + d - c = 2(s - c),$$

$$a + c + d - b = 2(s - b), b + c + d - a = 2(s - a).$$

$$\therefore \sin C = \frac{\sqrt{2(s - d) \cdot 2(s - c) \cdot 2(s - b) \cdot 2(s - a)}}{2(ab + cd)}$$

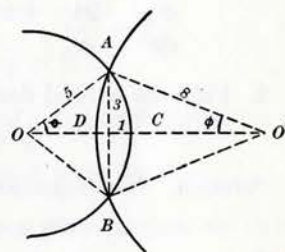
$$= \frac{2\sqrt{(s - a)(s - b)(s - c)(s - d)}}{ab + cd}.$$

$\therefore$ area of $ACBD$

$$= \frac{1}{2}(ab + cd) \cdot \frac{2\sqrt{(s - a)(s - b)(s - c)(s - d)}}{ab + cd}$$

$$= \sqrt{(s - a)(s - b)(s - c)(s - d)}.$$

22. Ex. Two circles whose radii are 5 and 8 inches respectively, intersect each other, so that the line connecting the points of intersection is 6 in. in length. What is the area included between the two intercepted arcs?



Solution.

$$\text{Area of sector } AOB = \frac{1}{2}(5)^2(2\theta) = \frac{25}{2}(2\theta).$$

$$\text{Area } \triangle AOB = \frac{1}{2}(5)^2 \sin 2\theta = \frac{25}{2} \sin 2\theta.$$

$$\therefore \text{area segment } ACB = \frac{25}{2}(2\theta - \sin 2\theta).$$

$$\text{Area of sector } AO'B = \frac{1}{2}(8)^2(2\phi) = 32(2\phi).$$

$$\text{Area } \triangle AO'B = \frac{1}{2}(8)^2(\sin 2\phi) = 32 \sin 2\phi.$$

$$\therefore \text{area segment } ADB = 32(2\phi - \sin 2\phi).$$

$$\text{But } \sin \theta = \frac{3}{5}. \quad \therefore \theta = 36^\circ 52' \text{ and } 2\theta = 73^\circ 44'.$$

$$\text{and } \sin \phi = \frac{3}{8}. \quad \therefore \phi = 22^\circ 2' \text{ and } 2\phi = 44^\circ 4'.$$

$$\begin{aligned} \therefore \text{area} &= \frac{25}{2}(2\theta - \sin 2\theta) + 32(2\phi - \sin 2\phi) \\ &= \frac{25}{2}(1.2868 - .96) + 32(.76911 - .6955) \\ &= 6.44 \text{ sq. in.} \end{aligned}$$

DIFFERENTIAL CALCULUS

1. Find the fifth derivative of
- $x^4 \log x$
- .

Solution. Let $y = x^4 \log x$.

$$\text{Then } \frac{dy}{dx} = 4x^3 \log x + x^3.$$

$$\frac{d^2y}{dx^2} = 12x^2 \log x + 4x^2 + 3x^2 = 12x^2 \log x + 7x^2.$$

$$\frac{d^3y}{dx^3} = 24x \log x + 12x + 14x = 24x \log x + 26x.$$

$$\frac{d^4y}{dx^4} = 24 \log x + 24 + 26.$$

$$\frac{d^5y}{dx^5} = \frac{24}{x}.$$

2. Find the second derivative of

$$y = \sin^{-1} x.$$

Solution. The equation may be written

$$\sin y = x.$$

$$\text{Then } \cos y \, dy = dx, \text{ or } \frac{dy}{dx} = \frac{1}{\cos y}.$$

$$\frac{d^2y}{dx^2} = \frac{\sin y}{\cos^2 y}.$$

$$\text{Since } \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2},$$

$$\therefore \frac{d^2y}{dx^2} = \frac{x}{(1 - x^2)^{\frac{3}{2}}}.$$

3. Find the derivative of
- $y = \log \frac{x}{a^x}$
- .

Solution. $dy = d(\log x - x \log a).$

$$\frac{dy}{dx} = \frac{1}{x} - \log a.$$

4. Find the derivative with respect to
- x
- of

$$y = \log \frac{1 + \sqrt{x}}{1 - \sqrt{x}}.$$

Solution. The equation may be written thus,

$$\log(1 + \sqrt{x}) - \log(1 - \sqrt{x}).$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1 + \sqrt{x}} + \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1 - \sqrt{x}} \\ &= \frac{x^{-\frac{1}{2}}}{1 - x}. \end{aligned}$$

5. If
- $y = a \cos(\log x) + b \sin(\log x)$
- , eliminate the constants
- a
- and
- b
- and obtain the equation

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

Solution. Taking the derivative of y and multiplying by x , we have

$$x \frac{dy}{dx} = -a \sin \log x + b \cos \log x.$$

Applying this process to both sides of the equation, we have

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} = -(a \cos \log x + b \sin \log x) = -y.$$

$$\therefore x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + y = 0.$$

6. Solve the equation

$$\arcsin 2x - \arcsin x = \frac{\pi}{3}.$$

Solution. Put $\arcsin 2x = \alpha$, and $\arcsin x = \beta$.

$$\text{Then } \alpha - \beta = \frac{\pi}{3} \quad (1)$$

$$\begin{aligned} \text{and } \sin \alpha &= 2x, \quad \cos \alpha = \sqrt{1 - 4x^2}, \\ \sin \beta &= x, \quad \cos \beta = \sqrt{1 - x^2}. \end{aligned}$$

Taking the cosine of each member of (1), we get

$$\sqrt{1-4x^2} \sqrt{1-x^2} + 2x^2 = \frac{1}{2}.$$

Isolating the radical, squaring and collecting terms

$$3x^2 = \frac{3}{4}.$$

$$\therefore x = \pm \frac{1}{2}.$$

Substituting in the given equation, we find $x = \frac{1}{2}$ is the only value.

7. Water is flowing into a conical reservoir 20 ft. deep and 10 ft. across the top at the rate of 15 cu. ft. per minute. Find how fast the surface is rising when the water is 8 ft. deep.

Solution.

$$\text{Volume of water} = V = \frac{1}{3} \pi r^2 h.$$

By similar triangles, $\frac{r}{h} = \frac{5}{20},$

or

$$r = \frac{h}{4}.$$

Hence

$$V = \frac{\pi h^3}{48}.$$

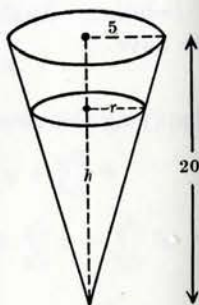
$$dV = \frac{\pi h^2}{16} dh.$$

$$\frac{dV}{dt} = \frac{\pi h^2}{16} \cdot \frac{dh}{dt}.$$

But we have given that $\frac{dV}{dt} = 15.$

$$\therefore \frac{\pi h^2}{16} \cdot \frac{dh}{dt} = 15 \text{ and } \frac{dh}{dt} = \frac{240}{\pi h^2}$$

when $h = 8, \frac{dh}{dt} = \frac{15}{4\pi} = 1.19 \text{ ft. per minute.}$



INTEGRAL CALCULUS

$$+ 1. \int \frac{x^3 - 4x^2 + 1}{(x-2)^2} dx.$$

$$\begin{aligned} \text{Solution. } \int \frac{x^3 - 4x^2 + 1}{(x-2)^2} dx &= \int \left[x + \frac{1-4x}{(x-2)^2} \right] dx \\ &= \int \left[x + \frac{A}{x-2} + \frac{B}{(x-2)^2} \right] dx \\ &= \int \left[x - \frac{x}{x-2} + \frac{-7}{(x-2)^2} \right] dx \\ &= \frac{x^2}{2} - 4 \log(x-2) + \frac{7}{x-2} + C. \end{aligned}$$

$$+ 2. \int \frac{x^3 - 2x^2 + 7x + 4}{(x^2 - 1)^2} dx.$$

$$\begin{aligned} \text{Solution. } \int \frac{x^3 - 2x^2 + 7x + 4}{(x^2 - 1)^2} dx \\ = \int \left(\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} \right) dx \end{aligned}$$

[Solving for A, B, C, and D, we have
 $x^3 - 2x^2 + 7x + 4 = A(x^3 + x^2 - x - 1) + B(x^2 + 2x + 1) + C(x^3 - x^2 - x + 1) + D(x^2 - 2x + 1).$

Now equating coefficients of like powers, we get

$$1 = A + C.$$

$$-2 = A + B - C + D.$$

$$4 = -A + B + C + D.$$

$$\therefore A = -1, B = \frac{5}{2}, C = 2 \text{ and } D = -\frac{3}{2}.]$$

$$\begin{aligned} + &= \int \left(-\frac{1}{x-1} + \frac{5}{2(x-1)^2} + \frac{2}{x+1} - \frac{3}{2(x+1)^2} \right) dx \\ &= \int -\frac{1}{x-1} dx + \frac{5}{2} \int \frac{1}{(x-1)^2} dx + 2 \int \frac{1}{x+1} dx \\ &\quad - \frac{3}{2} \int \frac{1}{(x+1)^2} dx \end{aligned}$$

$$\begin{aligned}
 &= 2 \log (x+1) - \log (x-1) - \frac{5}{2(x-1)} + \frac{3}{2(x+1)} + C \\
 &= \log \frac{(x+1)^2}{x-1} + \frac{-5x-5+3x-3}{2(x^2-1)} + C \\
 &= \log \frac{(x+1)^2}{(x-1)} + \frac{-x-4}{x^2-1} + C = \log \frac{(x+1)^2}{x-1} + \frac{x+4}{1-x^2} + C.
 \end{aligned}$$

$$3. \int \frac{x^3+1}{x^4-3x^3+3x^2-x} dx.$$

$$\begin{aligned}
 \text{Solution. } \int \frac{x^3+1}{x^4-3x^3+3x^2-x} dx &= \int \frac{x^3+1}{x(x-1)^3} dx \\
 &= \int \left(\frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{(x-1)^3} \right) dx. \\
 [\text{Solving for } A, B, C, \text{ and } D \text{ we have}] \\
 &= \int -\frac{1}{x} + \frac{2}{x-1} + \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} dx \\
 &= -\log x + 2 \log (x-1) - \frac{1}{x-1} + \frac{1}{(x-1)^2} + C \\
 &= \log \frac{(x-1)^2}{x} + \frac{1-x+1+C}{(x-1)^2} \\
 &= \log \frac{(x-1)^2}{x} - \frac{x}{(x-1)^2} + C.
 \end{aligned}$$

$$4. \int \frac{x^7 dx}{(9x^4-3)^{\frac{3}{4}}}.$$

$$\text{Solution. Let } \sqrt[4]{9x^4-3} = r, \text{ Then } dx = \frac{r^3}{9x^3} dr.$$

$$\begin{aligned}
 \int \frac{x^7 dx}{(9x^4-3)^{\frac{3}{4}}} &= \int \frac{x^7(r^3) dr}{9x^3(r^3)} = \int \frac{x^4 dr}{9} = \frac{1}{81} \int (r^4+3) dr \\
 &= \frac{1}{81} \left(\frac{r^5}{5} + 3r + C \right) = \frac{1}{81} \left(\frac{r^4}{5} + 3 \right) r + C \\
 &= \frac{1}{135} (3x^4+4)(9x^4-3)^{\frac{1}{4}} + C.
 \end{aligned}$$

$$+ 5. \int \frac{d\theta}{2+\sin \theta}.$$

$$\begin{aligned}
 \text{Solution. } \int \frac{d\theta}{2+\sin \theta} &= \int \frac{(2-\sin \theta)d\theta}{4-\sin^2 \theta} = 2 \int \frac{d\theta}{4-\sin^2 \theta} - \int \frac{\sin \theta d\theta}{4-\sin^2 \theta} \\
 &= 2 \int \frac{d\theta}{3+\cos^2 \theta} - \int \frac{\sin \theta d\theta}{3+\cos^2 \theta} \\
 &= 2 \int \frac{\sec^2 \theta d\theta}{3 \sec^2 \theta + 1} + \int \frac{d(\cos \theta)}{3+\cos^2 \theta} \\
 &= 2 \int \frac{\sec^2 \theta d\theta}{4+3 \tan^2 \theta} + \int \frac{d(\cos \theta)}{3+\cos^2 \theta} \\
 &= \frac{2}{\sqrt{3}} \int \frac{d(\sqrt{3} \tan \theta)}{4+3 \tan^2 \theta} + \int \frac{d(\cos \theta)}{3+\cos^2 \theta} \\
 &= \frac{2}{\sqrt{3}} \cdot \frac{1}{2} \tan^{-1}(\sqrt{3} \tan \theta) + \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\cos \theta}{\sqrt{3}} \right) \\
 &= \frac{1}{\sqrt{3}} \left[\tan^{-1}(\sqrt{3} \tan \theta) + \tan^{-1} \left(\frac{\cos \theta}{\sqrt{3}} \right) \right].
 \end{aligned}$$

$$+ 6. \int_{-2}^{-1} \frac{dx}{\sqrt{x^2-8x}}.$$

$$\text{Solution. Let } r = \sqrt{x^2-8x}.$$

$$\text{Then } 2r dr = (2x-8) dx.$$

$$\begin{aligned}
 \int_{-2}^{-1} \frac{dx}{\sqrt{x^2-8x}} &= \int_{-2}^{-1} \frac{2r dr}{(2x-8)r} = \int_{-2}^{-1} \frac{dr}{x-4} \\
 &= \int_{-2}^{-1} \frac{dr}{\sqrt{r^2+16}} = [\log (r+\sqrt{r^2+16})]_{-2}^{-1} \\
 &= [\log \sqrt{x^2-8x} + \sqrt{x^2-8x+16}]_{-2}^{-1} \\
 &= [\log \sqrt{x^2-8x} + x-4]_{-2}^{-1} \\
 &= \log (-2) - \log (\sqrt{20}-6) \\
 &= \log (-2) - \log (-2)(3-\sqrt{5}) \\
 &= -\log (3-\sqrt{5}).
 \end{aligned}$$

7. Integrate $\int \frac{4x^6 - a^3}{\sqrt{(x^6 - a^3)}} dx$.

Solution. Let $x^2 = z$. Then $dx = \frac{dz}{2\sqrt{z}}$.

$$\begin{aligned}\therefore \int \frac{4x^6 - a^3}{\sqrt{(x^6 - a^3)}} &= \int \frac{4z^3 - a^3}{2\sqrt{(z^4 - za^3)}} dz \\ &= \sqrt{(z^4 - a^3z)} + C = x\sqrt{(x^6 - a^3)} + C.\end{aligned}$$

8.* Find the length of the arc of the cissoid

$$\rho = 2a \tan \theta \sin \theta,$$

$$\text{from } \theta = 0 \text{ to } \theta = \frac{\pi}{4}.$$

The formula for the length of the arc gives

$$\begin{aligned}S &= 2a \int_0^{\frac{\pi}{4}} (4 + \tan^2 \theta)^{\frac{1}{2}} \tan \theta d\theta \\ &= 2a \int_1^{\sqrt{2}} \frac{(3 + y^2)^{\frac{1}{2}}}{y} dy, \quad y = \sec \theta, \\ &= 2a \left[\sqrt{5} - 2 - \sqrt{3} \log \frac{(\sqrt{3} + \sqrt{5})(2 + \sqrt{3})}{\sqrt{2}} \right].\end{aligned}$$

PHYSICS

1. Two lights, one 9 times as strong as the other, are 64 ft. apart. At what point between them will an object receive the same light from each of them?

Solution. Let x = distance from the weaker light, and $64 - x$ = distance from the stronger.

Since "the intensity of light diminishes as the square of the distance from its source increases," we have

$$(64 - x)^2 : 9 = x^2 : 1.$$

Solving, $x = 16$ or -32 . Hence the object is between the lights and 16 ft. from the weaker light (or it may be 32 ft. from the weaker light and 96 ft. from the stronger).

*From *American Mathematical Monthly*.

2. A lady stands 8 ft. from a vertical mirror, and should it be moved 7 ft. farther away, what distance would she be from her image then?

Solution. The image seems as far back of a plane mirror as the object is in front of it.

$$\therefore \text{distance lady is from her image} = 2(8 \text{ ft.} + 7 \text{ ft.}), \text{ or } 30 \text{ ft.}$$

3. Two men, A and B, have respectively 216 and 180 pounds luggage. They decide to place it together and carry it on a pole 11 ft. long. Where shall they place the luggage so that each shall carry his own?

Solution. Let x = distance from one having 216 pounds. Then taking moments about point where luggage is placed,

$$x \times 216 = 180(11 - x).$$

$$\text{Solving, } x = 5 \text{ ft.}$$

4. What power, acting horizontally at the center of a wheel $4\frac{1}{2}$ ft. in diameter and weighing 270 pounds, will draw it over a cylindrical log 6 in. in diameter, lying on a horizontal plane?

Solution. Let A denote the center of the log and B that of the wheel.

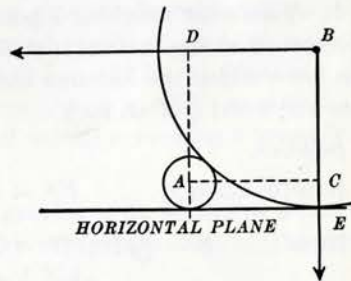
In $\triangle ADB$

$$AD = 24 \text{ inches}$$

$$AB = 30 \text{ inches}$$

$$DB = \sqrt{(30^2 - 24^2)},$$

or 18 inches



Now if x denotes the horizontal force applied at B and acting in the direction BD , and $w = 270$ lb., the weight of the wheel, x and w will be in equilibrium when

$$x : w = DB : BC, \text{ or } x : 270 = 18 : 24$$

Solving, $x = 202\frac{1}{2}$ lb. Hence any force greater than $202\frac{1}{2}$ lb. will draw the wheel over the log.

5. What temperature will result from mixing 100 lb. of ice at 14° with 80 lb. of steam at 270° F.?

Solution. The latent heat of ice is .505 and that of steam is .48. Heat of fusion of ice is 80 and heat of vaporization of water 537.

Then,

$$100 \left[\frac{5}{8} (32 - 14) \times .505 + 80 + \frac{5}{8} \text{ of } 180 \right] = 18,505,$$

the number of heat units necessary to melt the ice and raise the temperature of the resulting water to 212° F.

$$80 \left[\frac{5}{8} (270 - 212) \times .48 + 537 \right] = 44,197\frac{1}{3}$$

the number of heat units set free in reducing steam from 270° F. to water at 212° .

$$(44,197\frac{1}{3} - 18,505) \div 537 = 47.844,$$

the number of pounds of steam at 212° F. remaining.

The result is therefore 47.844 lbs. steam at 212° F. and 132.16 lbs. water at 212° F.

6. A hammer weighing 2 pounds has a velocity of 12 feet per second at the instant it strikes the head of a nail. Find the force which the hammer exerts on the nail if it is driven into the wood $\frac{1}{4}$ of an inch.

Solution.

$$\text{Kinetic Energy,} \quad FS = \frac{WV^2}{2g}.$$

$$\text{Here} \quad S = \frac{1}{48} \text{ ft., } W = 2 \text{ pounds, } V = 12 \text{ ft.}$$

$$\text{Then} \quad F \left(\frac{1}{48} \right) = \frac{2(12)^2}{2(32)}.$$

$$\therefore F = 216 \text{ pounds.}$$

7. A uniform bar, 3 ft. long and weighing 5 lbs., can turn freely about a point 11 inches from one end, and from that end a weight of 15 lbs. is suspended. From what point of

the bar must a weight of 30 lbs. be suspended so as to preserve equilibrium?

Solution. Taking moments about the turning point of the bar, we have

$$(15 \times 11) + \left(\frac{1}{2} \times \frac{1}{8} \times 5 \right) - 30x - \left(\frac{2}{2} \times \frac{2}{8} \times 5 \right) = 0,$$

where x denotes the distance from the turning point at which the 30-lb. weight must be suspended.

Solving, $x = 4\frac{1}{3}$, the distance the weight must be suspended in inches from the turning point.

8. The weights of two balls are as 9 to 25, but the weights of a cubic foot of the metals of which these are made are 15 to 9. What is the ratio of their volumes?

Solution. Densities are as 15 to 9.

$$\therefore \text{volumes are as } \frac{9}{15} : \frac{25}{9} = \frac{27}{125}.$$

9. A ball thrown downward with a velocity of 35 feet per second reaches the earth in $12\frac{1}{2}$ seconds. (a) How far has it moved, and (b) what is its final velocity?

Solution. During that time, gravity alone moved it 2512.5 ft. The additional force moved it $(35 \times 12\frac{1}{2})$, or 437 $\frac{1}{2}$ ft. Together they moved it $(2512.5 + 437.5)$, or 2950 ft. Its final velocity due to gravity is (32.16×12.5) , or 402 ft., to which we must add the initial velocity, making a velocity of 437 ft.

10. A ball whose specific gravity is $3\frac{3}{8}$ measures a foot in diameter. Find the diameter of another ball of the same weight but a specific gravity of $2\frac{1}{2}$.

Solution. Weight of first ball is $3\frac{3}{8}$ times an equal bulk of water, while that of second is $2\frac{1}{2}$ times the equal bulk of water. Hence $3\frac{3}{8}$ times volume of first = $2\frac{1}{2}$ times volume of second ball. Since the volumes vary as the cubes of the diameters,

$$\therefore d = \sqrt[3]{3\frac{3}{8} \div 2\frac{1}{2}}, \text{ or } 1\frac{1}{8} \text{ ft.}$$

11. Find the specific gravity of a solid which weighs 100 grams in air and 64 grams in a liquid whose density is 1.2.

Solution. The liquid displaced by the solid weighs $100 - 64$, or 36 grams. An equal bulk of water would weigh as many grams as 1.2 is contained in 36, which is 30 grams.

$\therefore$ specific gravity of the solid = $100 \div 30$, or $3\frac{1}{3}$.

12. To find the specific gravity of a wooden sphere which sinks $\frac{3}{4}$ of its diameter under water.

Solution. Assume diameter to be 4 ft.

Height of segment out of water = 1 ft.

Volume of segment = $\frac{\pi}{3}(3R - h)h^2$, or $\frac{5}{3}\pi$.

Volume of sphere = $\frac{32}{3}\pi$

$\frac{32}{3}\pi - \frac{5}{3}\pi = \frac{27}{3}\pi$, or 9π . (Vol. of segment submerged)

Specific gravity = $\frac{\text{Vol. of water displaced}}{\text{Vol. of sphere}}$

$$= 9 \div \frac{32}{3} = .84\frac{1}{2}.$$

NOTE. This is true of any sphere which sinks $\frac{3}{4}$ of its diameter.

13. On a false balance 8 cubes on the short arm balance 3 oz. One cube on the long arm balances 6 oz. What is the correct weight of the 8 cubes?

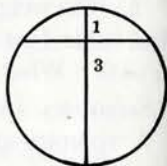
Solution. Let x = weight of a cube in oz.

y = length of short arm.

z = length of long arm.

Then $8xy = 3z$

and $xz = 6y$.



Multiplying the equations by z and $8y$, respectively, we have

$$8xyz = 3z^2$$

and $8xyz = 48y^2$

$$\therefore 48y^2 = 3z^2, \text{ or } z = 4y.$$

Then 3 oz. on long arm = 4×3 oz., or 12 oz., on short arm. $\therefore$ each cube weighs $\frac{1}{3}$ of 12 oz., or $1\frac{1}{2}$ ounces.

14.* Will a boat sunk in midocean ever reach the bottom? If it does reach the bottom, what principles of physics are involved? If it does not reach the bottom, state why.

Solution. The boat will reach the bottom if its specific gravity exceeds that of the sea water at the depth. Water being only slightly compressible, its specific gravity is always less than 2. At a depth of two miles the specific gravity of sea water is about 1.4. The principle is that of Archimedes, namely, that any body will sink in a liquid of lower specific gravity because it displaces a weight of liquid less than its own weight. Hence, there is a resultant force downward.

15.* What is the relation between the velocity of a man and the velocity of the end of his shadow, l being the height of the source of light and h being the height of the man?

Solution. Let S = horizontal distance of end of the shadow from the source of light, and s = distance of man from the source. Then from similar triangles, $\frac{S}{s} = \frac{l}{(l-h)}$.

Since $S = Vt$ and $s = vt$, the ratio of the displacements of the man and of the shadow is constant and equal to $\frac{S}{s}$.

Hence the ratio of the velocities is $\frac{l}{(l-h)}$.

* From "School Science and Mathematics."

16.* At what distance from a 40-candle-power mantle burner would a newspaper receive the same illumination as it would receive from an 8-candle-power incandescent lamp 2 feet distant from it?

$$\begin{aligned}\text{Solution. } 40 : 8 &= x^2 : (2)^2 \\ 8x^2 &= 160 \\ x^2 &= 20 \\ x &= 4.47 \text{ feet.}\end{aligned}$$

17. Assuming that a bomb explodes immediately upon striking the ground, calculate the time that elapses from the instant it is released 1000 feet above the ground until the sound of the explosion returns to that level.

Solution. The time required for the bomb to drop 1000 ft. under the influence of gravity is $\sqrt{\left[\frac{(2 \times 1000)}{32.16}\right]} = 7.88 \text{ sec.}$, the time taken by the sound to reach the point of release from the ground is $\frac{1000}{1090} = .91 \text{ sec.}$ Hence the time that will elapse between the release of the bomb and the sound of the explosion at the point of release is $7.88 + .91 \text{ sec.} = 8.79 \text{ sec.}$

18. A clock having a pendulum 60 cm. long keeps correct time. The pendulum is lengthened to 60.5 cm. How many seconds will the clock lose per day?

Solution. Denoting periods of clocks with pendulums 60 cm. and 60.5 cm. long by T_1 and T_2 , respectively, we have

$$\frac{T_1}{T_2} = \sqrt{\frac{60 \text{ cm.}}{60.5 \text{ cm.}}}$$

The time recorded by a clock varies inversely as the period. If the clock of period T_1 records 86,400 seconds in a day, the clock with period T_2 will record $86,400 \sqrt{\frac{60}{60.5}}$ sec. per day.

* From "School Science and Mathematics."

Hence time lost by clock of period T_2 is

$$86,400 \left(1 - \sqrt{\frac{60}{60.5}}\right), \text{ or } 362 \text{ seconds per day.}$$

19. A glass ball, whose cubical coefficient of expansion is .000024, weighs in air 90 gm. If it is put into a certain liquid the temperature of which is 12° , it weighs then 49.6 gm. If the liquid is heated to 97° , the weight of the glass ball will be 51.9 gm. What is the coefficient of expansion of the liquid?

Solution.

The volume of the ball is v_1 and v_2 at 12° and 97° .

The specific gravity of the liquid is s_1 and s_2 at 12° and 97° .

Call the expansion coefficient for the liquid x and for the glass g .

$$\text{Then } v_1 s_1 = 90 - 49.6 = 40.4,$$

$$v_2 s_2 = 90 - 51.9 = 38.1,$$

$$\text{but } v_2 s_2 = \frac{v_0 s_0 (1 + 97g)}{1 + 97x}$$

$$\text{and } v_1 s_1 = \frac{v_0 s_0 (1 + 12g)}{1 + 12x}.$$

$$\text{Hence } \frac{1 + 97g}{1 + 12g} \times \frac{1 + 12x}{1 + 97x} = \frac{38.1}{40.4}$$

$$\frac{1 + 97x}{1 + 12x} = \frac{40.4}{38.1} \times \frac{1.002328}{1.000288} = 1.0625.$$

$$\therefore x = 0.0007418,$$

the coefficient of expansion for the liquid.

20. Find the weight of a sphere of radius r , which floats in a liquid of specific gravity 3, with $\frac{1}{4}$ of its surface above the surface of the liquid.

Solution.

$$\text{Area of zone} = 2\pi rh.$$

$$\frac{1}{4} \text{ surface of sphere} = \pi r^2 = 2\pi rh.$$

$$\therefore h = \frac{1}{2} r.$$

The volume of portion above the liquid

$$\begin{aligned} &= \pi h^2 \left(r - \frac{h}{3} \right) \\ &= \frac{\pi r^2}{4} \left(r - \frac{r}{6} \right) \\ &= \frac{\pi r^2}{4} \cdot \frac{5r}{6} = \frac{5\pi r^3}{24} \end{aligned}$$

Since volume of sphere = $\frac{4}{3}\pi r^3$, the volume of the portion immersed = $\frac{4}{3}\pi r^3 - \frac{5}{24}\pi r^3$, or $\frac{9}{8}\pi r^3$.

Since the weight of a floating body is equal to the weight of the liquid displaced,

Volume of the liquid displaced = $\frac{9}{8}\pi r^3$.

$\therefore$ weight of sphere = $\frac{9}{8}\pi r^3 s$.

21. The nozzle of a fire hose has an opening 2 in. in diameter while the pipe just back of it is 3 in. in diameter. Find the pressure just back of the nozzle when it can throw a jet 60 ft. vertically upward.

Solution.

Let v_1 = veloc. in 3 in. tube.

v_2 = veloc. in 2 in. tube.

m = mass of water.

Neglecting atmospheric pressure,

1. $mgh + \frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2$.

2. $v_2^2 = 2 \times 32.2 \times 60 = 62.2$ ft. per sec.

3. $v_1 = \left(\frac{2}{3}\right)^2 \times 62.2 = 27.64$.

4. Substituting in 1,

$$gh + 382 = 1,932.$$

$$\therefore h = 48^+.$$

$$\therefore \text{pressure} = \frac{4.8}{144} \times 62.4 = 20.8^+ \text{ lbs. per sq. in.}$$

Since this pressure is opposed to atmospheric pressure, the absolute pressure = $20.8^+ + 14.7 = 35.5^+$ lbs. per sq. in.

22. If a burner can heat 2 kg. of water from 10° to 80° C. in 10 minutes how much water can it boil away into steam in one hour?

Solution. Heat necessary to vaporize 1 gm. of water = 536 cal.; to vaporize 1 kgm. of water, 536,000 cal.

Burner gives heat at rate of $2000 \times 70 \div 10 = 14,000$ calories per min.

In 60 min. $60 \times 14,000 = 840,000$ cal.

$840,000 \div 536,000 = 1.567$ kgm. of water which would be vaporized.

23. The pitch of a screw is 2 cm. and the lever is 3 m. long. If the efficiency of the machine is 20 per cent, how much can a man lift with this screw if he exerts a pull of 50 kgm. on the end of the lever?

Solution. The mechanical advantage of the screw is

$$\frac{\frac{2}{2} \times \frac{r}{300} \times \frac{\pi}{3.1416}}{2(\text{pitch})} = 942.48.$$

24. From the basket of a balloon hangs a spring balance. This balance carries a mass of 100 grams. The balloon ascends with an acceleration of 200 C. G. S. units. What will be the apparent weight of the 100 grams during the ascent?

Solution. The total acceleration during the ascent is $980 + 220 = 1200$ C. G. S. units. The spring balance interprets an acceleration of 980 as one gram, hence it will interpret an acceleration of 1200 as 1.224 gm. and it would then read 122.4 grams as the apparent weight of 100 gm.

25. A piece of iron of density 7.5 floats in mercury (density = 13.5) and is completely covered with water which rests on the top of the mercury. How much of the iron is immersed in the mercury?

Solution. Consider the volume to be 1 cu. cm.

Let x = part in mercury, and $1 - x$ = volume in water.

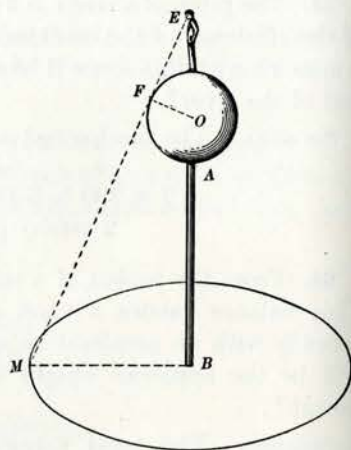
Then x times $13.5 + (1 - x) 1 = 7.5$ by Archimedes' principle.

$\therefore x = .52$, the part in mercury, and $1 - x = .48$, the part in water.

MISCELLANEOUS

1. A ball 12 ft. in diameter rests on the top of a pole 60 ft. high. On the ball stands a man whose eyes are 6 ft. above the ball. What is the area of the ground invisible to him?

Solution. Let AB denote the pole, B being the base; O the center of the ball; E the position of the man's eye. Draw EFM tangent to the ball at F and touching the ground at M . Draw OF and BM . Then BM is the radius of the circle whose area is invisible. $\triangle EFO$ and EBM are similar rt. $\triangle$ having $\angle BEM$ common.



$$EF = \sqrt{EO^2 - OF^2} = \sqrt{12^2 - 6^2} = \sqrt{108} = 6\sqrt{3}.$$

$$\text{Then } \frac{BM}{OF} = \frac{EB}{EF}, \text{ or } \frac{BM}{6} = \frac{78}{6\sqrt{3}}.$$

$$\therefore \text{area of ground invisible} = \pi(26\sqrt{3})^2, \text{ or } 6371.16 \text{ sq. ft.}$$

2. A 20-inch ball is placed in the corner of a room on the floor. What must be the diameter of another ball which can be placed back of this, both balls touching floor, walls, and each other?

Solution. Suppose a ball to be inscribed in a cube whose edge is one inch. Place the cube in the corner and make a vertical section, $CFLD$, through the diagonal of the cube which meets the corner of the room. This section is a rectangle whose width is 1 and length

$$\sqrt{(1^2 + 1^2)}, \text{ or } \sqrt{2}.$$

The diagonal of rectangle or of cube is

$$\sqrt{(1^2 + (\sqrt{2})^2)} = \sqrt{3}.$$

The centers of the balls, O, O' are all on this diagonal. OT being $\frac{1}{2}$, TF is $\frac{1}{2}\sqrt{3} - \frac{1}{2}$ or $\frac{1}{2}(\sqrt{3} - 1)$, and if we represent $O'P$ by r , OF is $\frac{1}{2}(\sqrt{3} - 1) - r$. Then by similar right triangles,

$$r : 1 = \frac{1}{2}(\sqrt{3} - 1) - r : \sqrt{3}.$$

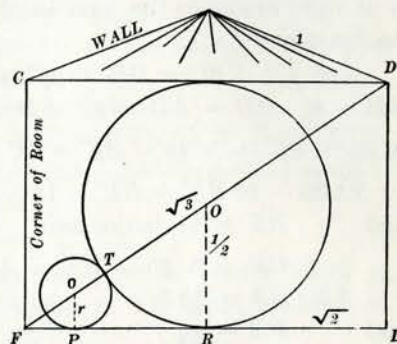
$$\text{Whence } r\sqrt{3} = \frac{1}{2}(\sqrt{3} - 1) - r$$

$$\text{or } r\sqrt{3} + r = \frac{1}{2}(\sqrt{3} - 1).$$

$$\text{Factoring, } r(\sqrt{3} + 1) = \frac{1}{2}(\sqrt{3} - 1)$$

$$\text{or } r = \frac{\frac{1}{2}(\sqrt{3} - 1)}{\sqrt{3} + 1} = \frac{\frac{1}{2}(\sqrt{3} - 1)}{\sqrt{3} + 1} \times \frac{\sqrt{3} - 1}{\sqrt{3} - 1} = \frac{2 - \sqrt{3}}{2}$$

and $2r = 2 - \sqrt{3}$, which is the diameter of small ball, when the large one is one inch in diameter; hence $20(2 - \sqrt{3}) = 5.35898$ inches, the required diameter. An infinite series of balls may be thus placed whose diameters are $20, 20(2 - \sqrt{3}), 20(2 - \sqrt{3})^2, 20(2 - \sqrt{3})^3 \dots$ to infinity. Balls placed in the other direction have for their diameters $20, 20(2 + \sqrt{3}), 20(2 + \sqrt{3})^2, 20(2 + \sqrt{3})^3$ to infinity.



3. If I should drink from a full conical cup, depth 4 inches and diameter of top 6 inches, until the surface of the water is at right angles to the slant height, how many cubic inches do I consume?

Solution. $BC = AC = 5$, since
 $AB = 6$, $BD = AD = 3$, $CD = 4$.

$$\sqrt{36 - BE^2} + \sqrt{25 - BE^2} = AC = 5.$$

$$\therefore \sqrt{900 - 61BE^2} + BE^4 = 18 - BE^2$$

and $BE = \frac{24}{5}$, major axis.

$$CE = \sqrt{25 - BE^2} = \frac{7}{5},$$

$$FE : BA = \frac{7}{5} : 5,$$

and $FE = \frac{42}{25}$.

O is center of elliptic section of water.

$$\frac{1}{4}(\text{minor axis})^2 = HO \times OK;$$

$$\therefore \text{minor axis} = \sqrt{AB \times FE} = \frac{6}{5}\sqrt{7}.$$

$$\text{Vol. } ABC = 12\pi.$$

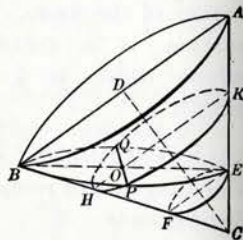
$$\text{Vol. } CBE = \frac{\pi}{4} \times \frac{24}{5} \times \frac{6\sqrt{7}}{5} \times \frac{7}{15} = \frac{84\pi\sqrt{7}}{125};$$

$$\therefore 12\pi - \frac{84\pi\sqrt{7}}{125} = 32.1136 \text{ cu. in.}$$

4. Let a cube be inscribed in a sphere, a second sphere in this cube, and so on. Find the diameter of the 7th sphere, if that of the first is 27 inches. What would be the volumes of all the spheres so inscribed, including the first?

Solution. Let d represent the diameter of first sphere, or diagonal of inscribed cube whose edge is $\sqrt{\left(\frac{d^2}{3}\right)} = \frac{d}{\sqrt{3}}$ and also diameter of second sphere; $\frac{d}{\sqrt{3^2}}$ is diameter of the third sphere, and $\frac{d}{\sqrt{3^{n-1}}}$ is the diameter of the n th sphere.

$$\therefore d \div \sqrt{3^6} = 1, \text{ the diameter of 7th sphere.}$$



Side of inscribed cube is $\sqrt{(d^2 \div 3)} = d \div \sqrt{3}$, diameter of 2d sphere; $d \div \sqrt{3^2}$ is the diameter of 3rd sphere, $d \div \sqrt{3^{n-1}}$ = diameter of n th sphere.

$\therefore d \div \sqrt{3^6} = 1$, diameter of 7th sphere. The diameters are easily found to be 27, $9\sqrt{3}$, 9, $3\sqrt{3}$, 3, $\sqrt{3}$, 1.

The total volume is found as follows:

$$\begin{aligned} T.V. &= \frac{1}{6}\pi(27^3 + (9\sqrt{3})^3 + 9^3 + (3\sqrt{3})^3 + 3^3 + (\sqrt{3})^3 + 1) \\ &= \frac{1}{6}\pi(19,683 + 2187\sqrt{3} + 729 + 81\sqrt{3} + 27 + 3\sqrt{3} + 1) \\ &= \frac{1}{6}\pi(20,440 + 2271\sqrt{3}) \\ &= 12761.9 \text{ cubic inches.} \end{aligned}$$

5. In how many ways can a set of twelve black and twelve white pieces be placed on the black squares of a checker board?

Solution. There are 32 black squares in a checker board, 12 of which will have white men, 12 black men, and 8 will be blank. Then by the well-known principles of permutations, we have,

$$\frac{1 \times 2 \times 3 \times 4 \times 5 \times \dots \times 32}{(1 \times 2 \times 3 \times \dots \times 12)(1 \times 2 \times 3 \times \dots \times 12)(1 \times 2 \times \dots \times 8)} = 28,443,124,054,800 \text{ ways.}$$

6. Two balls, diameters 8 and 13, rest upon a plane and are in contact with each other. What is the diameter of a ball that will just pass between them?

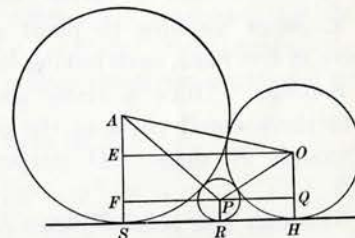
Solution. Let $AS = 6\frac{1}{2}$; $OH = 4$; and $OE, QF \parallel$ to HS ; then $AE = 2\frac{1}{2}$ and $AO = 10\frac{1}{2}$; hence

$$\begin{aligned} QF &= \sqrt{(10\frac{1}{2})^2 - (2\frac{1}{2})^2} \\ &= 2\sqrt{26}. \end{aligned}$$

Let $RP = r$; then $AP = 6\frac{1}{2} + r$ and $AF = 6\frac{1}{2} - r$;

$$FP = \sqrt{(6\frac{1}{2} + r)^2 - (6\frac{1}{2} - r)^2} = \sqrt{26r}.$$

$$OP = 4 + r \text{ and } OQ = 4 - r; \text{ hence}$$



$$PQ = \sqrt{(4+r)^2} - (4-r)^2 = \sqrt{16r} \text{ and}$$

$$QF = \sqrt{26r} + \sqrt{16r}; \text{ but } QF = 2\sqrt{26}.$$

$$\therefore \sqrt{26r} + \sqrt{16r} = 2\sqrt{26} \text{ and } r = 1.256; \text{ diameter, } 2.512.$$

7. A square park contains 40 acres. It is required to lay off another park containing the same area, inclosed by an iron fence forming circular arcs only and to find the cost of the fence at \$2.00 per rod.

Solution. Let $ABCD$ denote the square park. Then the side $AB = \sqrt{40 \times 160}$, or 80 rd.

With radius OA describe a circle. Also with the same radius describe the arcs APB and BRC .

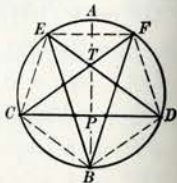
Then the area of the pelecoid $APBRC$ is bounded by circular arcs only and has an area equal to that of the square park. Also the length of the required fence equals the circumference of the circumscribing circle = 80π , or 251.33 rd.

$$\therefore \text{cost of fence} = 251.33 \times \$2, \text{ or } \$502.66.$$

8. Show me how to plant a symmetrical group of ten trees in five rows, each having four trees.

Solution. Draw a circle and divide its diameter, AB , into three equal parts at the points P , T . Through P draw CD perpendicular to AB .

Through the point T draw DE and CF . Also join BE and BF . Plant the trees at the vertices of this inscribed pentagon and at the intersections of its diagonals.



9. Find the area included between the parabola $x^2 = 4ay$ and the witch

$$y = \frac{8a^3}{x^2 + 4a^2}.$$

Solution. To determine the limits of integration we solve the equations simultaneously to find where the curves intersect. The coordinates of A are found to be $(-2a, a)$, and of $C(2a, a)$.

It is seen from the figure that area $AOCB$ = area $DECBA$ - area $DECOA$.

But

$$\text{area } DECBA = 2 \times \text{area } OECB = 2 \int_0^{2a} \frac{8a^3 dx}{x^2 + 4a^2} = 2\pi a^2,$$

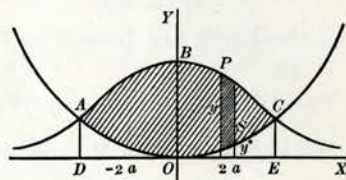
and

$$\text{area } DECOA = 2 \times \text{area } OEC = 2 \int_0^{2a} \frac{x^2}{4a} dx = \frac{4a^2}{3}.$$

$$\therefore \text{area } AOCB = 2\pi a^2 - \frac{4a^2}{3} = 2a^2\left(\pi - \frac{2}{3}\right).$$

Another method is to consider the strip PS as an element of the area. If y' is the ordinate corresponding to the witch, and y'' to the parabola, the differential expression for the area of the strip PS equals $(y' - y'')dx$. Substituting the values of y' and y'' in terms of x from the given equations, we get

$$\begin{aligned} \text{area } AOCB &= 2 \times \text{area } OCB \\ &= 2 \int_0^{2a} (y' - y'') dx \\ &= 2 \int_0^{2a} \left(\frac{8a^3}{x^2 + 4a^2} - \frac{x^2}{4a} \right) dx \\ &= 2a^2\left(\pi - \frac{2}{3}\right). \end{aligned}$$



10. Calculate the work done in pumping out the water filling a hemispherical reservoir 10 feet deep.

Solution. The equation of the circle is $x^2 + y^2 = 100$.

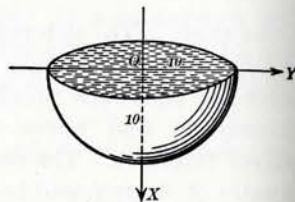
Hence $y^2 = 100 - x^2$,

$$W = 62,$$

and the limits are from $x = 0$ to $x = 10$.

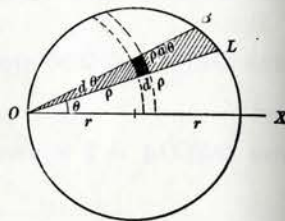
Substituting in formula, we get

$$\begin{aligned} \text{work} &= 62\pi \int_0^{10} (100 - x^2)xdx \\ &= 155,000\pi \text{ ft. lb.} \end{aligned}$$



11. Find the area of the circle $\rho = 2r \cos \theta$ by double integration.

Solution. Summing up all the elements in a sector (as OBL), the limits are 0 and $2r \cos \theta$; and summing up all such sectors, the limits are 0 and $\frac{\pi}{2}$ for the semi-circle OXB . Substituting in formula,



$$\frac{A}{2} = \int_0^{\frac{\pi}{2}} \int_0^{2r \cos \theta} \rho d\rho d\theta = \frac{\pi r^2}{2}, \text{ or } A = \pi r^2.$$

12. Find the area of the surface of revolution generated by revolving the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ about the axis of X .

Solution. Here $\frac{dy}{dx} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$, $y = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}$.

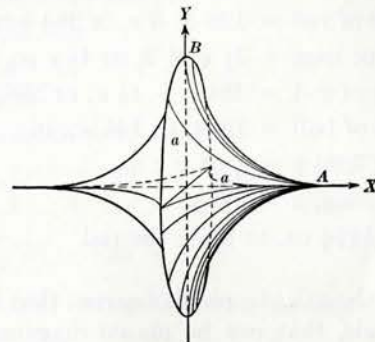
Substituting formula and noting that the arc BA generates only one half of the surface, we get

$$\frac{S_x}{2} = 2\pi \int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} \left[1 + \frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}} \right]^{\frac{1}{2}} dx$$

$$= 2\pi \int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} \left(\frac{a^{\frac{2}{3}}}{x^{\frac{2}{3}}} \right)^{\frac{1}{2}} dx$$

$$= 2\pi a^{\frac{1}{3}} \int_0^a (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}} x^{-\frac{1}{3}} dx = \frac{6\pi a^2}{5}.$$

$$\therefore S_x = \frac{12\pi a^2}{5}.$$



HOW MUCH SILVER IS REQUIRED?

13. A blacksmith had an iron ball,
Which he did a bullet call;
He found out, by some nice trick,
It was exactly one foot thick;
And silvered o'er with plate,
And which he proved by scales and weight;
The silver's weight he did pronounce
Was just exactly half an ounce;
He took the ball and made it hot,
And forged there out an iron rod;
The rod was just three inches thick,
Round and long, just like a stick;
Now how much silver will it take
To plate the rod that he did make,
And put it on exact as thick
As that which on the ball did stick?

Solution.

Volume of ball = volume of cylindrical rod
 $= 12^3 \cdot \frac{1}{6} \pi$, or 288π cu. in.

Area of end of rod $= (\frac{3}{2})^2 \pi = 2\frac{1}{4} \pi$ sq. in.

$\therefore$ length of rod $= 288 \pi \div 2\frac{1}{4} \pi$, or 128 in.

Convex surface of rod $= 128 \times 3 \pi$, or 384π sq. in.

Area of base $= 2\frac{1}{4} \pi \times 2$, or $4\frac{1}{2} \pi$ sq. in.

Total surface of rod $= 384 \pi + 4\frac{1}{2} \pi$, or $388\frac{1}{2} \pi$ sq. in.

Surface of ball $= 12^2 \pi$, or 144 sq. in.

$\therefore 144 \pi : 388\frac{1}{2} \pi = \frac{1}{2} \text{ oz.} : x \text{ oz.}$

Solving, $x = 1\frac{2}{3}\frac{1}{8}$.

$\therefore$ it will take $1\frac{2}{3}\frac{1}{8}$ oz. to plate the rod.

14. Find the length of a piece of carpet that is a yard wide, with square ends, that can be placed diagonally in a room 40 ft. long and 30 ft. wide, the corners of the carpet just touching the walls.

Solution. Let $ABCD$ denote the room and $GFEH$ the strip of carpet.

Let $x = EC = AG$

and $y = CF = AH$.

$\triangle ECF$ and HDE are similar.

$\therefore 40 - x : 30 - y = y : x$

or $40x - x^2 = 30y - y^2$. (1)

Also $x^2 + y^2 = 9$

or $x = \sqrt{9 - y^2}$.

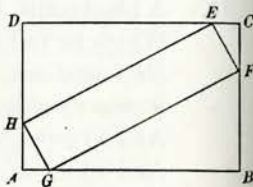
Substituting the value of x in (1) and reducing,

$$y^4 - 30y^3 + 616y^2 + 135y - 3579.75 = 0.$$

Solving by Horner's method,

$$y = 2.4337. \therefore x = 1.7541.$$

$\therefore$ length of carpet $= \sqrt{(40 - x)^2 + (30 - y)^2}$, or 47.144 ft.



15. Find the altitude of the cylinder of maximum volume that can be inscribed in a right cone.

Solution. Let $AC = r$ and $BC = h$.

Then volume of cylinder $= \pi xy^2$.

$\triangle ABC$ and DBG are similar.

$\therefore r : y = h : h - x$.

$$\therefore y = \frac{r(h - x)}{h}.$$

Hence the function to be tested is

$$V = \pi x \left[\frac{r(h - x)}{h} \right]^2 = \frac{\pi r^2}{h^2} [h^2x - 2hx^2 + x^3]. \quad (1)$$

$$\frac{dV}{dx} = \frac{\pi r^2}{h^2} [h^2 - 4hx + 3x^2]. \quad (2)$$

Placing $h^2 - 4hx + 3x^2 = 0$ and solving for x , we find the critical values of x are $\frac{h}{3}$ and $\frac{h}{3}$ being the maximum and h the minimum.

$\therefore$ altitude of maximum cylinder $= \frac{1}{3}$ altitude of cone.

16. Find the volume and radius of base of the maximum cylinder that can be inscribed in a cone 10 ft. high and base 4 ft. in diameter.

Solution. Substituting $\frac{h}{3}$ for x in (1) in previous exercise, we have

Volume of maximum cylinder $= \frac{4}{27} \pi r^2 h$.

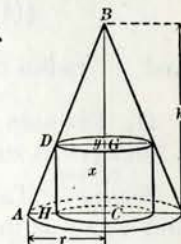
Volume of given cone $= \frac{1}{3} \pi r^2 h$.

$\therefore$ volume of maximum cylinder $= \frac{4}{9}$ volume of the cone.

From $y = \frac{r(h - x)}{h}$ in previous exercise by substituting $\frac{h}{3}$

for x , we have $y = \frac{2}{3} r$.

$\therefore$ radius of base of maximum cylinder $= \frac{2}{3}$ radius of base of cone.



$\therefore$ volume of maximum cylinder =

$$\frac{4}{9}(4\pi \cdot \frac{16}{3}) = \frac{160\pi}{27}, \text{ or } 18.62 \text{ cu. ft.}$$

and radius of base = $\frac{2}{3}$ of 2, or $1\frac{1}{3}$ ft.

17. The axes of two equal right circular cylinders of radius a intersect at right angles. Find their common volume.

Solution. Take the x -axis and y -axis as the axes of the cylinders. Then the volume of $OMBC$ is one eighth of the required volume.

Passing a plane perpendicular to the y -axis, the section, $OMBC$, is formed. A side of this section or square is

$$Q'P = QQ' = \sqrt{a^2 - y^2}.$$

Then area of this section is $Q'P \cdot Q'Q = a^2 - y^2$.

$$\text{Hence } \frac{V}{8} = \int_0^a x dy = \int_0^a (a^2 - y^2) dy = \frac{2a^3}{3}.$$

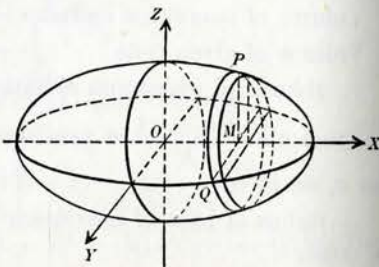
$$\therefore V = \frac{16}{3} a^3.$$

18. Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution. The section perpendicular to the x -axis at the distance x from the center O is an ellipse.

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 - \frac{x^2}{a^2}.$$



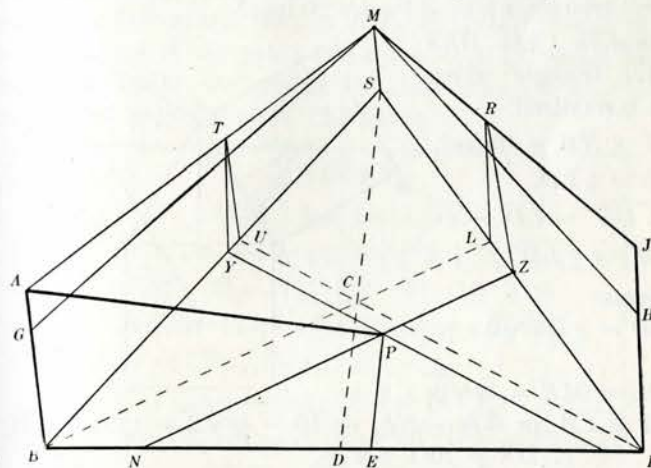
Its semi-axes are $PM = c \sqrt{1 - \frac{x^2}{a^2}}$

and $MQ = b \sqrt{1 - \frac{x^2}{a^2}}.$

Also its area is $\pi \cdot MP \cdot MQ = \pi bc \left(1 - \frac{x^2}{a^2}\right).$

$$\text{Hence } V = \int_{-a}^{+a} x dx = \int_{-a}^{+a} \pi bc \left(1 - \frac{x^2}{a^2}\right) dx = \frac{4}{3} \pi a bc.$$

19. Find length of pole based at a point within an equilateral triangle whose side is 200 feet, that will just reach the top of any of the three towers, standing upright at the corners, heights 60, 80, and 100 feet, respectively.



Solution. Let $BF = FS = BS = 200$ feet; $AB = 80$ feet, $FJ = 100$ feet, and $MS = 60$ feet. Denote the pole by AP . Draw HM parallel to $FS = 200$ feet; then $JH = 40$ feet. $BD = DF = SL = FL = 100$ feet. Draw RZ perpendicular to JM at its mid-point; then

$$HM : LR = HJ : LZ, \text{ whence } LZ = 16 \text{ feet.}$$

In a similar way find $UY = 7$ and $DE = 9$ feet. Draw NZ parallel to BL ; then $BN = 2LZ = 32$ feet, because $BF = 2FL$; hence $NE = 109 - 32 = 77$ feet. The pole will have its base at P , the intersection of the three perpendiculars erected at E , Y , and Z .

Let C be the center of the $\triangle BFS$; then $CD = \frac{100}{3}\sqrt{3}$ feet. $\triangle BDC$ and NEP are similar, and $BD:NE = DC:EP$; whence $EP = \frac{77}{3}\sqrt{3}$ feet; $EP^2 + BE^2 = BP^2$. Join BP ; but $AP^2 = BP^2 + AB^2$; hence $AP = \sqrt{[(\frac{77}{3}\sqrt{3})^2 + 109^2 + 80^2]} = 142.32$ feet.

20. What is the area of the triangle formed by joining the centers of the squares constructed on the sides of an equilateral triangle whose sides are 20 feet?

Solution. Let DEF be the triangle whose area is required.

$$DN = NB = 10 \text{ feet.}$$

$$NL = \frac{1}{2} LD.$$

$$\therefore \overline{DN}^2 = \overline{LD}^2 - \frac{1}{4} \overline{LD}^2, \text{ or } \frac{3}{4} \overline{LD}^2.$$

Hence

$$LD = \frac{2}{3} DN\sqrt{3}$$

and

$$DL = ME = \frac{20}{3}\sqrt{3},$$

$$LM = LB, \text{ or } NB - NL, \text{ or } 10 - \frac{10}{3}\sqrt{3} = \frac{10}{3}(3 - \sqrt{3}).$$

$$\therefore DE = 10(1 + \sqrt{3}),$$

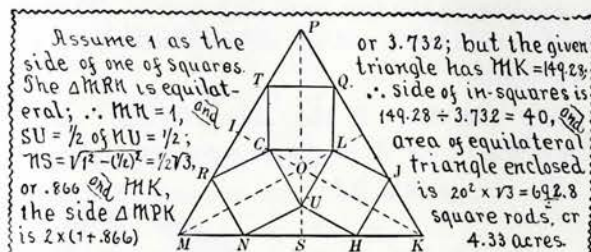
and area $\triangle DEF = \frac{1}{4}$ of $100(1 + \sqrt{3})^2\sqrt{3} = 50(2 + \sqrt{3})\sqrt{3}$ or 323.205 square feet.

—SOLVED BY G. B. M. ZEBB.*

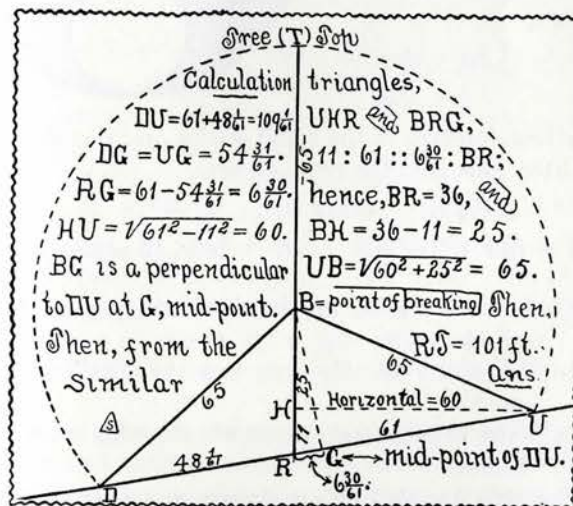
21. A triangular field measures 149.28 rods on each side. If 3 equal square lots are laid off in it, so that two corners of each lie in two adjacent sides of the field, and the other two

* See preface to "Mathematical Wrinkles."

corners coincide with two corners of the other lots, how many acres in central space enclosed by the 3 lots? * *Solution:*



22. On a hillside which slopes 11 feet in 61 feet of its length stands an upright pole. If this pole should break at a certain point, and fall up hill, the top would strike the ground 61 feet from the base of the pole; but if it should fall down hill, its top would strike the ground $48\frac{1}{4}$ feet from the base of the pole. Get length of pole. * *Solution:*



* Reproduced from "The School Visitor."

23. Two equal circles, largest possible, are inscribed in a circle whose circumference is 640 feet; then two other circles, largest possible, in the two remaining spaces. Find circumference of a small circle and the area not included in the four inscribed circles.

Solution. Let $BE = R$, the radius of circle; then AE is $\frac{1}{2}R$, the radius of one of the larger inscribed circles.

Let $r = BF$, radius of small circle.

Then $AF = \frac{1}{2}R + r$

and $EF = R - r$.

In the right triangle, AEF ,

$$\overline{AF}^2 - \overline{AE}^2 = \overline{EF}^2.$$

Substituting equivalents for these lines,

$$(\frac{1}{2}R + r)^2 - (\frac{1}{2}R)^2 = (R - r)^2.$$

$$\therefore r = \frac{1}{3}R.$$

The circumferences of the small circles are, therefore, $\frac{1}{3}$ of 640, or $213\frac{1}{3}$ rods, and the required area,

$$\pi R^2 - 2\pi(\frac{1}{2}R)^2 - 2\pi(\frac{1}{3}R)^2 = \frac{5}{18}\pi R^2.$$

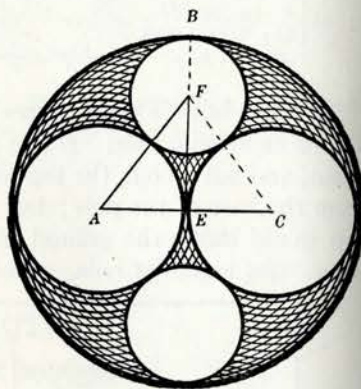
But $R = 640 \div 2\pi$, and $\frac{5}{18}\pi R^2 = 9054.13$ square feet.

24. There was a quaker Sands who for three sons divided off his lands;

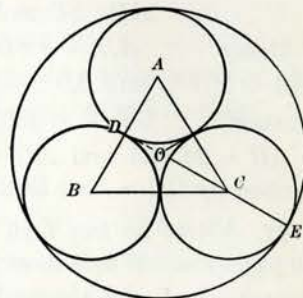
In the center of this acre was the dwelling of the quaker;

In center of each circle round, the dwelling of a son was found.

Compute by skill or art, how many rods they lived apart.

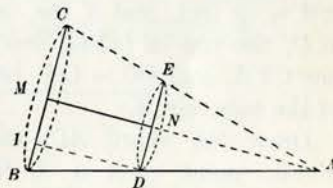


Solution. Let O be the center of the farm and A, B, C the centers of the sons' shares. Suppose AB to be 2; BD is then 1 and CD is 1.732; CO is $\frac{2}{3}$ of 1.732, or 1.154666, and EO is 2.154666. But EO is the square root of (160 divided by 3.14159265), which is 7.1365. Then $2.154666 : 2 = 7.1365 : 6.624$ rods, distance the sons live apart; and $\frac{1}{3}$ of 6.624 times 1.732, or 3.824 rods is the distance each son has to go to the father's house.



25. Two wheels are fastened to the ends of an axle 6 feet long. If the diameters of the wheels are 5 feet and 8 feet, how many revolutions will they make in turning once about a circle?

Solution. The wheels will revolve on the circumferences of concentric circles whose center is A , and radii are AB and AD , respectively. We have DI parallel to MN ; hence $DI = 6$ and $BI = 1\frac{1}{2}$. By similar triangles, $BI : DI = BM : AM$, or $1\frac{1}{2} : 6 = 4 : 16$; consequently, $AM = 16$. AB , the hypotenuse of ABM , is $\sqrt{(16^2 + 4^2)} = 4\sqrt{17}$, radius of path of larger wheel, and the path is $8\pi\sqrt{17}$; but the circumference of wheel is 8π , and it makes $8\pi\sqrt{17} \div 8\pi = \sqrt{17} = 4.1231$ revolutions, and smaller wheel the same.



26. A log 3 feet in diameter at one end and 2 feet at the other is rolled until it completes a circle; thus passing over 320π square feet of surface. Find circumference of circle described by end of log.

Solution. Using diagram in ex. 25, we have

$CB = 3$ feet, $ED = 2$ feet, and $BI = BM - DN$, or $\frac{1}{2}$ foot.

$$AD : BD = DN : BM = 1 : \frac{1}{2}.$$

Hence $AD = 2 BD$ and $AB = 3 BD$

and $\pi(3 BD)^2 - \pi(2 BD)^2 = 320 \pi$,

whence $5 \overline{BD}^2 = 320$, and $BD = 8$ feet.

$AB = 24$ feet and $AD = 16$ feet, and circumference described by $CB = 48 \pi$ feet, and that by $ED = 32 \pi$ feet.

27. A tree 80 feet high stands on the bank of a stream 40 feet wide. Where must it break off so that it may remain connected at the place of breaking, and the top may just touch the opposite shore?

Solution. Let $BC = 80$ feet and $AB = 40$ feet, and if the tree break at O , the top in falling describes the arc CYA , and $OC = OA$, being radii of the same circle.

Draw the chord AC , and at its middle point erect a $\perp HO$. The equal $\triangle AHO$ and CHO are similar to $\triangle ABC$, because they have the angle C in common.

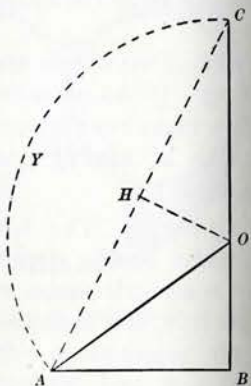
$\overline{AC}^2 = 80^2 + 40^2 = 8000$, and since AH is $\frac{1}{2} AC$ and the square on all the line is 4 times the square on half the line, then

$$\overline{AH}^2 = \frac{1}{4} \text{ of } 8000 = 2000.$$

By similar $\triangle$, $\overline{BC}^2 : \overline{AH}^2 = \overline{AC}^2 : \overline{AO}^2$, or $6400 : 2000 = 8000 : \overline{AO}^2$; whence $AO = 50$ feet and $BO = 30$ feet.

28. In how many ways can a clown make up a three-piece suit, if he has 8 coats, 4 vests and 10 pairs of trousers?

Solution. The suit could be made up in $8 \times 4 \times 10$, or 320 ways.



29. How many possible arrangements are there for 15 books on a shelf?

Solution. $15! = 1,307,674,368,000$.

30. Given a field 100 ft. square. Find the areas of the strips $MRSN$ and $ACBDEF$, each strip being 4 ft. wide.

Solution. (a) Let $MN = x$

and $EM = 100 - x$.

$$\therefore MR = \sqrt{100^2 + (100 - x)^2}.$$

Then area of strip $MRSN = 4\sqrt{100^2 + (100 - x)^2}$.

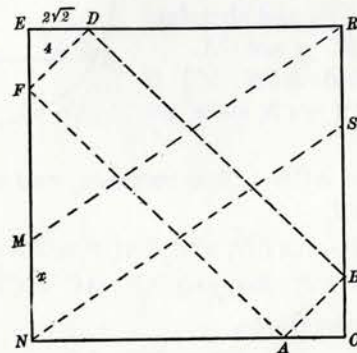
Also area of strip $MRSN = 100x$

(x being base and 100 the altitude of the rectangle).

$$\therefore 100x = 4\sqrt{100^2 + (100 - x)^2}.$$

$$\therefore 78x^2 + 25x = 2500. \quad \therefore x = 5.503.$$

$$\therefore \text{area of strip } MRSN = 100x = 550.3 \text{ sq. ft.}$$



Solution. (b) Let $DB = y$.

Since DEF is an isosceles rt. triangle,

then $2 \overline{ED}^2 = 4^2$, or $\overline{ED}^2 = 8$.

$$\therefore ED, \text{ or } EF = \sqrt{8} = 2\sqrt{2}.$$

In $\triangle DRB$, $DR = RB = 100 - 2\sqrt{2}$.

Then $\overline{DR}^2 + \overline{RB}^2 = \overline{DB}^2$.

$$\therefore 2(100 - 2\sqrt{2})^2 = y^2, \text{ or } y = (100 - 2\sqrt{2})\sqrt{2}.$$

Then area of strip $ACBDEF$

$$= 2(\text{area } \triangle DEF) + \text{area rectangle } BF$$

$$= (2\sqrt{2})^2 + 4\sqrt{2}(100 - 2\sqrt{2})$$

$$= 8 + 400\sqrt{2} - 16 = 400\sqrt{2} - 8, \text{ or } 557.68 \text{ sq. ft.}$$

31. If the vertices of an equilateral triangle inscribed in a circle are joined to any point on the circumference, one join equals the sum of the other two joins.

Solution. Ptolemy's Theorem gives an immediate solution of this problem, but the following proof is more elementary.

$AB = BC = CA$, and therefore arc $AB = \text{arc } BC = \text{arc } CA$.

In $\triangle ABP$ and $\triangle ABH$, $\angle 1$ is common, and $\angle 2 = \angle 3$, since arc $AB = \text{arc } CA$.

In $\triangle ACP$ and $\triangle ACH$, $\angle 4$ is common, and $\angle 5 = \angle 6$ since arc $CA = \text{arc } AB$.

Hence, $\triangle ABP \sim \triangle ABH$, and $\triangle ACP \sim \triangle ACH$.

$$\therefore AP : AB = BP : BH, \text{ and } AP : AC = CP : CH.$$

From these proportions,

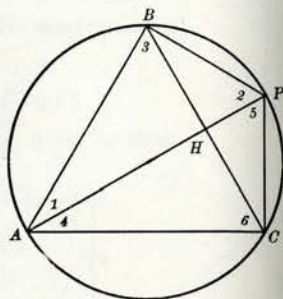
$$AP \cdot BH = BP \cdot AB$$

$$AP \cdot CH = CP \cdot AC.$$

$$\text{Adding, } AP(BH + CH) = BP \cdot AB + CP \cdot AC.$$

$$\text{But } BH + CH = BC = CA = AB.$$

$$\therefore AP = BP + CP.$$



KERNELS

NUT KERNELS

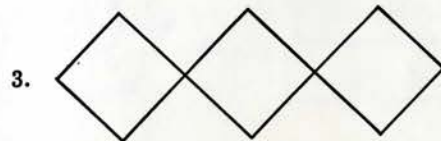
NUTS FOR YOUNG AND OLD

1. In 1 hour A can count $\frac{1}{5}$
 In 1 hour B can count $\frac{1}{11}$ } of the sum.
 In 1 hour A and B can count $\frac{16}{55}$

$\therefore$ in 7 hours they will count $7 \times \frac{16}{55}$, or $2\frac{2}{55}$ sums.

$$\therefore 2\frac{2}{55} - 1 = 1\frac{2}{55}.$$

2. 7. They may be easily counted by forming two tetrahedrons with a common base.



4. There are many such as $5\frac{5}{8}$, $7\frac{7}{8}$, etc.

5. None.

6. 36.

7. $44 + \frac{44}{4}$.

8. 123456789.

9. $[999]^{9-9}$; $.999\frac{9}{9}$; $.9\frac{9}{9}\frac{9}{9}$.

10. The same distance.

11. $11\frac{1}{4}$.

12. $\frac{1}{4}$ of 33, or $8\frac{1}{4}$.

13. $1 + 3 + 5 + 7 + \frac{7}{5} + \frac{3}{11}$.

14. When held before a looking glass, it reads thus:

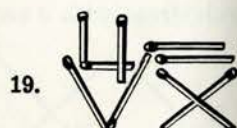
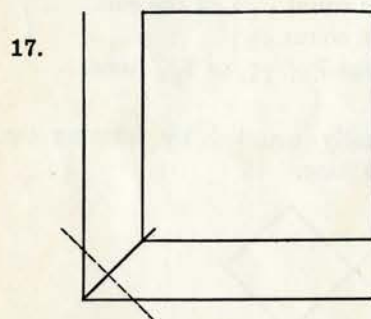
NINE
ONE
EIGHT

EIGHTEEN

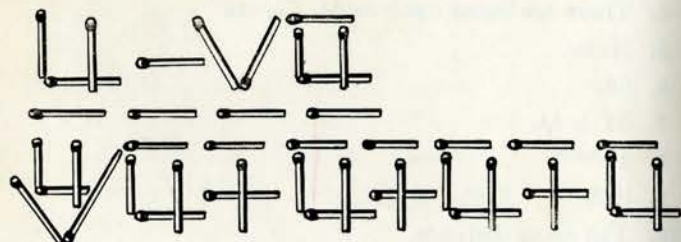
$$15. 100 = (5 + 5)(5 + 5) = \frac{5}{.5} \times \frac{5}{.5} = \frac{55}{.55} = \frac{55 - 5}{.5}$$

$$= 5\left(\frac{5+5}{.5}\right).$$

$$16. \frac{11}{VI} = \frac{2}{6} = \frac{1}{3}.$$



20.



Many other solutions are possible.

21. $999.99\frac{2}{3}$.

22. The lazy monkey went up.

23. $9 + \frac{9}{9}$.

24. 9 pounds weight of melon. 1 pound weight of slice.

25. Place 3 pieces in one cup and 7 in another, and then place one of these in the third cup.

26. $\frac{7}{18}$.

27. As each day and night the cat climbed up eleven feet, and came down seven, the daily upward gain was four feet, and thirteen days would bring her fifty-two feet up the tree. Then on the fourteenth day she mounted the remaining eleven feet, and was at the top, so that no coming down seven feet is to be taken into account, and she attains her place in fourteen days.

28. Since $492t04$ is a multiple of 9, t may be found by subtracting 19, the sum of the known digits, from 27, which gives $t = 8$.

29. The triangles do not all lie in the same plane.

30. None. If it was a hole, the dirt would not be there.

31. \$42.00 and the blankets.

32. $8[88888]^{8-8}$.

33. $3(\frac{3}{3})$; $\sqrt[3]{3^3}$.

35. 3 pounds.

36. Ten cents.

37. A difference of 25 cents in the price of wheat per bushel makes a difference of \$1 an acre in rent. Hence the rent paid in wheat is four bushels per acre. Deducting the value of 4 bushels of wheat from \$8, the total rent leaves \$4 an acre paid in cash. Then dividing 80 by 4 gives 20, the number of acres.

38. Three inches were lost on each yard in measuring eighteen feet. Nothing was lost in measuring the two feet with the 33-inch stick. *Ans.* 18 inches lost.

39. Each one made a tour of the world yearly for 50 years, one going the westerly route and the other the easterly.

40. In 12 hours it strikes

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12$,
or 78 times. In a day or 24 hours it strikes 2×78 , or 156 times.

41. The sum is 5×10 , or 50.

B's age = $\frac{1}{2}$ of $(50 - 10)$, or 20.

A's age = $20 + 10$, or 30.

42. 10 pigs.

43. 7.

44. $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$

$$\frac{1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45}{8 + 6 + 4 + 1 + 9 + 7 + 5 + 3 + 2 = 45}$$

45. $\frac{35}{70} + \frac{148}{296} = 1$.

46. 150%.

47. The statement is true although figures sometimes lie as follows $1000 \div 0 = 100000$.

48. Certainly you could, but you cannot multiply feet by feet and get square feet. If the product is square feet, the multiplicand must also be square feet. The multiplier is always an abstract number.

49. Explanation left to the reader.

50. If 366 of them have different birthdays (including Feb. 29 of a leap year), then the 367th one must duplicate one of these, for the entire list has been exhausted. If the 366 do not have different birthdays, then at least two must have the same one.

51. Two such solutions are

$$78\frac{3}{8} + 214\frac{5}{8} = 100; 50\frac{1}{2} + 49\frac{3}{8}$$

There are other solutions to this problem which is not a simple one.

52. He gave the customer two silver dollars, 45 cents in change and a $\$2\frac{1}{2}$ gold piece.

53. 0.

$$\begin{array}{r} 54. \quad 987654321 \\ 123456789 \\ \hline 864197532 \end{array}$$

There are other solutions, but this one is the easier to remember.

$$\begin{array}{r} 55. \quad 9876543210 \\ 1234567890 \\ \hline 8641975320 \end{array}$$

There are other solutions to this one also.

$$\begin{array}{r} 56. \quad 123456789 \\ 8 \\ \hline 987654312 \end{array}$$

57. 11 o'clock + 2 is 1 o'clock.

$$58. \quad \sqrt{\frac{1}{VT}} = 1$$

59. First number = $\frac{1}{3}$.

Second number = 3.

The product of the numbers = 1.

60. I gained \$1000.

61. This may be done as follows:

$$\begin{array}{r} 173 \quad 85 \\ 4 \quad 92 \\ \hline 177 \quad 177 \end{array}$$

62. 13 stamps cost $1\text{¢} + 25\text{¢}$, or 26¢ .

$\therefore$ 1 stamp cost 2¢ .

63. 219,438,657.

64. Let x = what 10 would be.

Then $10 : x = 4 : 6$

or $x = 15.$

65. One.

66. It is impossible.

67. Yes, by placing three on a table as vertices of an equilateral triangle and holding the fourth one so as to form a tetrahedron.

68. 15 cents.

69. A pint of each costs $3\frac{1}{3}$ cents.

$\therefore 1 \div 3\frac{1}{3}$, or $.3 =$ part of a pint of each brought home.

70. The hounds gain 5 rods on the hare in a unit of time.

$\therefore$ the time they run $= 96 \div 5$, or $19\frac{1}{5}$ units.

$\therefore$ distance the hounds ran before catching the hare $= 19\frac{1}{5} \times 21 \text{ rd.} = 403\frac{1}{5} \text{ rods.}$

71. None.

72. The amount A, B, and C had can be found by solving the following equations:

$$(a) \quad 2x - 14 = 10.$$

$$(b) \quad 2x - 16 = 10.$$

$$(c) \quad 3x - 35 = 10.$$

$\therefore$ A had \$12; B, \$13; and C, \$15.

$\therefore$ I borrowed \$2 from A, \$3 from B, and \$5 from C.

73. Let x = number of fish I caught.

Then, $\frac{1}{2}x - 2 =$ number fell back.

$$\frac{1}{6}x + 6 = \text{number in sack.}$$

$$\frac{1}{6}x - 5 = \text{number I fried.}$$

$$\frac{1}{3}x - 5 = \text{number the turtle got.}$$

$$\therefore \frac{7}{6}x - 6 = x.$$

$$\therefore x = 36.$$

NUTS FOR THE FIRESIDE

1. $\frac{9}{10}$ of a pound is the weight of $\frac{1}{10}$ of the melon. It therefore weighs 10 times $\frac{9}{10}$ lb. or 9 pounds.

2. Neither; but 6 and 7 are 13 and 6 and 7 is 13 are both correct.

3. The old \$10 bill is much better than a new "one" (dollar bill).

4. None.

$$5. 2 + \frac{2}{2} - \frac{2}{2} = 2.$$

6. 2.

7. The signs $\times$ and $\div$ take precedence over the signs $+$ and $-$; hence the operations of multiplication and division should always be performed before addition and subtraction.

8. $\frac{1}{2}$ of the weight of the goose was 10 lb. Its weight was therefore 20 lb.

9. Since 6 cats eat 1 rat in 1 min. only 6 cats will be required.

10. 3 ounces.

Let $x =$ weight of apple.

Then, $x = \frac{3}{4}x + \frac{3}{4}$. Solving $x = 3.$

11. 59 minutes. The sixtieth piece would not need to be cut.

12. 12 cents. Only three cuts are necessary. Cut each link of one of the pieces and connect the four other pieces.

13. $\frac{1}{2}$. $\frac{1}{2}$ yard square equals $\frac{1}{4}$ square yard and is $\frac{1}{2}$ of $\frac{1}{2}$ square yard.

14. 16 and 64.
 15. The bottle cost \$1.05 and the cork 5¢.
 16. No.
 17. The answer is B.
 18. Six dozen dozen = $6 \times 12 \times 12 = 864$, while one half a dozen dozen = $\frac{1}{2} \times 12 \times 12$, or 72.
 19. Take the goat over, return and take the cabbage over, bring the goat back, take the wolf over, then return for the goat.
 20. Give one of the boys the box with an orange in it.
 21. Only 8 cats.
 22. 3 ducks.
 23. 21 days, because two of the ears are his own.
 24. 199.
 25. 10.
 26. 14.
 27. $3\frac{2}{3}$, $5\frac{1}{7}$, $11\frac{5}{9}$, etc. There are many such numbers.
 28. Numbers like $99\frac{1}{4}$, or $99\frac{1}{7}$.
 29. At the beginning of the eighth day it is 3 ft. from the top and therefore at the end of that day reaches the top.
Ans. 8 days.
 30. 18 cents.
 31. Fill the 3-gallon cask and pour it into the 5. Fill it again and pour into the 5 until the 5 is full. There is now 1 gallon left in the three. Empty the 5 and pour the one in it. Fill the 3 again and pour into the 5 — making exactly 4 quarts.
 32.
$$\begin{array}{r} \text{Twice twenty five} = 50 \\ \text{Twice five and twenty} = 30 \\ \text{Difference} = 20 \end{array}$$

 33. $33 - 3$, $3^3 + 3$, $5(5) + 5$, or $6(6) - 6$.

34. None. The rest flew away.
 35. $111\frac{1}{2}$ yards.
 36. A pound of feathers. Gold is weighed by Troy weight in which a pound is lighter than in the avoirdupois weight. A Troy pound = 5760 grains while an avoirdupois pound = 7000 grains. A pound of feathers and a pound of lead weigh the same.
 37. The one who bought the pawn ticket which was good for only 25 cents (\$1 less 75).
 38. The first had 5, the second had 7.
 39. 56 quarts. Twenty-four quart bottles equals eighty quarts, from which deduct twenty-four quarts.
 40. Take VI and turn it upside down, making AI. On top of this put VI and you have XI.
 41. By drawing a horizontal line through the number half of it is found to be 000, or 0.
 42. 1, 8, 11, 69, 96, etc. There are many such numbers.
 43. Similar to # 40.
 44. XIX. Take 1 away and have XX.
 45. IX = 9. Cross the I and make it XX.
 46. The number appears as 13212.
 47. 300 lbs.
 48. The clown wearing the number 6 stands on his head and the result is exactly divisible by 9 for every possible arrangement.
 49. Weight = $10 + \frac{1}{2}(10) + \frac{1}{2}$ weight.
 $\therefore \frac{1}{2}$ weight = 15 lb. or, weight = 30 lb.
 50. An odd number.
 51. $13 = x + \frac{x}{x} = 12 + \frac{12}{12}$
 or $13 = x - \frac{x}{x} = 14 - \frac{14}{14}$

$$52. 17 = x + x + \frac{x}{x} = 8 + 8 + \frac{8}{8}$$

$$\text{or } 17 = x + x - \frac{x}{x} = 9 + 9 - \frac{9}{9}$$

$$53. 19 = x + \frac{x+x}{x+x} = 18 + \frac{18+18}{18+18}$$

$$\text{or } 19 = x - \frac{x+x}{x+x} = 20 - \frac{20+20}{20+20}$$

54. In 51, $x = 12$, or 14.

In 52, $x = 8$, or 9.

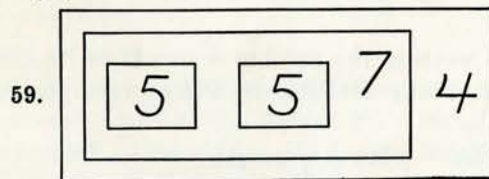
In 53, $x = 18$, or 20.

55. $\frac{9}{9}$. Turn the page upside down and the fraction is still $\frac{9}{9}$.

56. 12.

57. $\frac{1}{2}$
 $\frac{34}{56}$
 $\frac{7}{100}$

58. 45 is the number. She divided it into 4 parts, viz. 8, 12, 5, and 20.



60. Let x and y denote the fractions.

Then, $x - y = xy$, or $y = \frac{x}{x+1}$.

If $x = \frac{1}{2}$, then $y = \frac{1}{3}$.

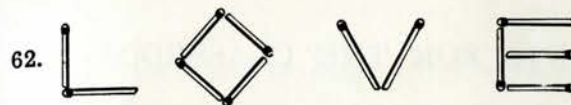
Or if $x = \frac{1}{3}$, then $y = \frac{1}{4}$.

There is an infinite number of such fractions.

61. They arranged themselves as follows:

1, 2, 7, 8 and 5, 3, 4, 6,

the boy wearing #9 standing on his head.



63. Form a tetrahedron.

$$64. [2 + 4 - 4]^0 - 1 = 2^0 - 1 = 1 - 1 = 0.$$

65. Let $x =$ what he had.

$$\text{Then } x + x + 2x + \frac{x}{2} - 9 = 0.$$

$$\therefore x = 2.$$

66. He placed it so that the base formed a square with A , B , C , and D as mid-points and the sides making $\angle$ of 45° with the sides of the first base.

67. 44.

68. Left for the reader.

69. 2 apples. He took one and left one.

70. 3 ducks.

71. Nothing.

72. Let $x =$ the number.

$$\text{Then, } \frac{x}{5} = x - 5.$$

$$\therefore x = 6\frac{1}{4}.$$

A CRACKED NUT

What number divided by itself is more than when multiplied by itself?

Solution: Any proper fraction.

For example, $\frac{1}{3} \div \frac{1}{3} = 1$,
 while $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

NUTS FOR THE CLASSROOM

1. Let d = diameter of circle described by the man's feet.
 $d + 12$ = diameter of circle described by the man's head.
 $\pi(d + 12)$ = distance his head moves.
 πd = distance his feet moves.
 Difference = $\pi(d + 12) - \pi d$, or 12π .

2. Let r = radius of earth.
 $2\pi r$ = length of the earth's equator.
 $r + x$ = radius if 48 inches is added to the equator.
 Then, $r : r + x = 2\pi r : 2\pi r + 48$.
 $\therefore x = \frac{24}{\pi} = 7.6^+$ in.

3. Let x = grandpa's age.
 $\frac{x}{3}$ = her age when $\frac{1}{3}$ of grandpa's age.
 $\frac{x}{3} - 5$ = increase in age of each.
 $x = 55 + \left(\frac{x}{3} - 5\right)$.
 $\therefore x = 75$.

4. Let x = price per dozen in cents.
 $6x$ = cost of 6 doz.
 $12\left(\frac{50}{x}\right)$ = number of eggs bought for 50 cents.
 Then, $6x = 12\left(\frac{50}{x}\right)$.
 $\therefore x = 10$.

5. Since his chance of drawing a ten-dollar bill is 1 out of 4, he should pay $\frac{1}{4}$ of \$25, or \$6.25.

6. 4 lbs.

7. 3 oz.

8. Take the sum of the digits from 9, or the next higher multiple of 9.

9. Let x = wife's age and $x + y$ = husband's age.

Then, $15\left(\frac{x}{2} - y\right) = \frac{3}{2}(x + 2y)$

and $x + y - 2 + x + y - 2 + y = 50$.

Simplifying, $x = 3y$

and $2x + 3y = 54$.

Solving, $x = 18$ and $y = 6$.

$\therefore$ wife's age is 18 and husband's is 24.

10. 40.

11. The difference between any two numbers with reverse digits is always 9 or some multiple of 9. Since B's age is 10 times C's age, the difference between A's age and B's age must be 9. Then C's age must be one half of 9, or $4\frac{1}{2}$ years; B's age is 10 times $4\frac{1}{2}$, or 45 years; A's age is 45 plus 9, or 54 years.

12. Let x = no. of apples.

$$x - \left(\frac{x}{2} + \frac{1}{2}\right) = \frac{x}{2} - \frac{1}{2} = \text{1st Rem.}$$

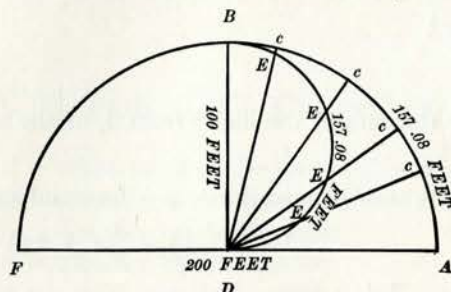
$$\frac{x}{2} - \frac{1}{2} - \left(\frac{x}{4} - \frac{1}{4} + \frac{1}{2}\right) = \frac{x}{4} - \frac{3}{4} = \text{2nd Rem.}$$

$$\frac{x}{4} - \frac{3}{4} - \left(\frac{x}{8} - \frac{3}{8} + \frac{1}{2}\right) = \frac{x}{8} - \frac{7}{8} = \text{3rd Rem.}$$

$$\therefore \frac{x}{8} - \frac{7}{8} = 1.$$

Solving, $x = 15$.

13. The dog follows the arc DEB , which is half the circumference of a circle whose radius is 100 ft., or 157.08 ft.



14. Let x = rate of Mary's digging. Then x also equals Ann's rate of pulling tops. Let y = rate of Ann's digging. Then $3y$ equals rate of Mary's pulling tops.

The rate of Ann's pulling tops is to the rate of Ann's digging as the rate of Mary's pulling tops is to the rate of Mary's digging.

$$\therefore x : y = 3y : x.$$

Then, $x^2 = 3y^2$ and $x = y\sqrt{3} = 1.732y$.

That is, Mary can dig 1.732 times as fast as Ann. The \$10 should therefore be divided in the ratio of 1.732 and 1. Mary should receive \$6.34 and Ann \$3.66.

15. Let x = number of nuts.

Then,

$$1 + \frac{1}{4}(x - 1) = \frac{x}{4} + \frac{3}{4} = \text{what 1st son received.}$$

$$x - \left(\frac{x}{4} + \frac{3}{4}\right) = \frac{3}{4}x - \frac{3}{4} = \text{remainder.}$$

$$1 + \frac{1}{4}\left(\frac{3}{4}x - \frac{3}{4} - 1\right) = \frac{3}{16}x + \frac{9}{16} = \text{what 1st daughter received.}$$

$$\frac{3}{4}x - \frac{3}{4} - \left(\frac{3}{16}x + \frac{9}{16}\right) = \frac{9}{16}x - \frac{21}{16} = \text{remainder.}$$

$$1 + \frac{1}{4}\left(\frac{9}{16}x - \frac{21}{16} - 1\right) = \frac{9}{64}x + \frac{27}{64} = \text{what 2nd son received.}$$

$$\frac{9}{16}x - \frac{21}{16} - \left(\frac{9}{64}x + \frac{27}{64}\right) = \frac{27}{64}x - \frac{111}{64} = \text{remainder.}$$

$$1 + \frac{1}{4}\left(\frac{27}{64}x - \frac{111}{64} - 1\right) = \frac{27}{256}x + \frac{81}{256} = \text{what 2nd daughter received.}$$

$$\begin{aligned} \therefore \left(\frac{x}{4} + \frac{3}{4}\right) + \left(\frac{9}{64}x + \frac{27}{64}\right) \\ = \left(\frac{3}{16}x + \frac{9}{16}\right) + \left(\frac{27}{256}x + \frac{81}{256}\right) + 100. \end{aligned}$$

Solving, $x = 1021$.

Children received 256; 192; 144; 108; and Mother 321.

16. x = age of 1st child.
 $x - 12$ = age of 9th child.
 $x - 15$ = age of 11th child.
 $x - 21$ = age of 15th child.

Then, $3x - 48 = x$

or $x = 24$.

$\therefore$ ages are 24, $22\frac{1}{2}$, 21, $19\frac{1}{2}$, 18, $16\frac{1}{2}$, 15, $13\frac{1}{2}$, 12, $10\frac{1}{2}$, 9, $7\frac{1}{2}$, 6, $4\frac{1}{2}$, and 3.

17. $\frac{10}{c}$ = direct ratio.

$\frac{c}{10}$ = inverse ratio.

Then, $\frac{c}{10} = 9\left(\frac{10}{c}\right)$

or $c = 30$.

Let m denote the multiplier.

$$\text{Then, } \frac{10}{30m} = 9\left(\frac{30m}{10}\right).$$

$$\therefore m = \frac{1}{9}.$$

18. In two hours from the time the boy left for the swim the minute hand would be at the same place that it was when he started and the hour hand would have moved $\frac{2}{15}$ of the distance around the dial.

Since the hands had changed places on his return, the time over two hours would be the time it would take for the combined movement of both hands to go $\frac{1}{15}$ of the distance around the dial. As the minute hand moves 12 times as fast as the hour hand it would move over $\frac{12}{13}$ of the distance, or $46\frac{2}{13}$ minute spaces. The time, therefore, exceeds the two-hour limit by $46\frac{2}{13}$ minutes.

19. The number must have been that multiple of seven which is one more than a common multiple, but not the least common multiple, of two, three, four, five, and six; or $2 \times 3 \times 4 \times 5 \times 6 + 1 = 721$, the number.

20. The L. C. M. of 4, 8, 12, and 16 = 48, the number of weeks.

Ans. 48 weeks. Dec. 4, 1923.

21. Let $x = DC$. Then $9 + x =$ radius of large circle and $18 + 2x =$ its diameter. $2(9 + x) - 9$, or $9 + 2x =$ diameter of small circle. Then $4\frac{1}{2} + x =$ radius of small circle.

Let M be the center of the small circle.

Then in rt. $\triangle ECM$, we have

$$\overline{EM}^2 = \overline{EC}^2 + \overline{CM}^2$$

or $(4\frac{1}{2} + x)^2 = (4 + x)^2 + (4\frac{1}{2})^2.$

Solving, $x = 16.$

$\therefore$ diameters of large and small circles are 50 and 41 respectively.

22. Let $x =$ Bill's rate per hour
and $y =$ Phil's rate per hour.

Then, $13\frac{1}{2}x =$ distance from the meeting point to Dallas.

$24y =$ distance from the meeting point to Wichita Falls.

Hence, $\frac{24y}{x} = \frac{13\frac{1}{2}x}{y} \quad (1)$

or $y = \frac{3}{4}x.$

Also $24y - 13\frac{1}{2}x = 18. \quad (2)$

Substituting value of y found in (2), we find $x = 4$ and $y = 3.$

$\therefore$ distance = $13\frac{1}{2}x + 24y = 126$ miles.

23. Let $x =$ the auto's age.

$y =$ the tires' age.

$x - y =$ the difference.

$y - (x - y) =$ tires' age when the auto was as old as the tires are now.

$y + (x - y) =$ age of tires when as old as the car is now.

Then, $x = 2[y - (x - y)], \quad (1)$

and $y + (x - y) + x + (x - y) = 2\frac{1}{4}. \quad (2)$

Solving (1) and (2), we find

$$x = 1 \text{ and } y = \frac{3}{4}.$$

$\therefore$ age of auto = 1 year.

24. The statement of this problem is the same as # 23 except (1) and (2) will read as follows:

$$x = 3[y - (x - y)] \quad (1)$$

and $y + (x - y) + x + (x - y) = 100. \quad (2)$

Solving, $x = 42\frac{2}{3}$ and $y = 28\frac{1}{3}.$

25. The problem in division was solved by the little boy wearing #6. He stood on his head so that the number 931 was divisible by 7.

26. Adding we find 7 cats and 7 kittens weigh 70 pounds.

$\therefore$ a cat and a kitten weigh $\frac{1}{7}$ of this amount, or 10 pounds.

$$4 \text{ cats} + 3 \text{ kittens weigh } 37 \text{ pounds.} \quad (1)$$

$$3 \text{ cats} + 4 \text{ kittens weigh } 33 \text{ pounds.} \quad (2)$$

Multiplying (2) by 4 and (1) by 3, we have

$$12 \text{ cats} + 16 \text{ kittens weigh } 132 \text{ pounds}$$

$$12 \text{ cats} + 9 \text{ kittens weigh } 111 \text{ pounds.}$$

Subtracting, 7 kittens weigh 21 pounds.

$\therefore$ a kitten weighs 3 pounds and a cat weighs 7 pounds.

27. Let $x =$ number of hills in each row.

$$\text{No. of hills hoed by B} = (x - 3) + 6 = x + 3$$

$$\text{No. of hills hoed by A} = 3 + (x - 6) = \frac{x - 3}{6}$$

$$\text{Difference} = \frac{x - 3}{6}$$

$\therefore$ B hoes 6 hills more than A.

28. If he paid \$3.00 to go from A to C and return, he would only have to pay \$1.50 to go from B to C and return. He therefore should charge only $\frac{1}{2}$ of \$1.50, or \$.75.

29. Let h = number of horses.
 c = number of chickens.

Then, number of heads = $h + c$,
 number of wings = $2c$,
 number of feet = $4h + 2c$.
 $\therefore h + c + 2c = 4h + 2c$

or $c = 3h$.

75% are chickens and 25% are horses.

30. You would never get out of the room however fast you walk. If, however, it takes 1 minute to walk the first distance, $\frac{1}{2}$ minute to walk the next distance, and so on, the number of minutes required to reach the door is $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$, etc. In algebra we learn that the sum of these figures does not exceed 2. You could therefore reach the door in 2 minutes.

31. In each case the sum is 1,083,676,269.

32. Let x = my age.
 Then, $35 + 21 + x - (54 + 4) = 2x - 20$.
 Solving, $x = 18$.

33. Let S denote the starting point. Then draw a line SA representing a distance of 24 miles. Let B' denote the mid-point and B the mid-point of $B'A$. Then SB = distance traveled by B , which obviously is 18 miles.

34. In traveling, I went 1 mile in $\frac{1}{20}$ hour. In returning, I went 1 mile in $\frac{1}{30}$ hour. In going and returning I averaged going 1 mile in $\frac{1}{2}$ of $(\frac{1}{20} + \frac{1}{30})$, or $\frac{1}{24}$ hr.

Therefore my average rate was 24 miles per hour.

35. When a number is divisible by 9, the sum of its digits must be a multiple of 9. Since the sum of the digits is 77, the missing figure is found by subtracting 77 from 81, the

higher multiple of 9. The missing figure is therefore 4, which may be placed before, after, or anywhere within the number and it will then be exactly divisible by 9.

36. 1, 1, 1, 1, 1, 3, 5, 7.

37. For any number selected the result is always 10890.

38. Let x = cost of keeping him.
 $26 + x$ = total cost.

$13 + \frac{x}{4}$ = loss.

Cost - loss = selling price.

$\therefore (26 + x) - (13 + \frac{x}{4}) = 60$

or $x = 62\frac{2}{3}$.

$\therefore$ loss = $13 + \frac{x}{4}$, or $\$28\frac{2}{3}$.

39. The answer is the L. C. M. of 2, 3, 4, 5, 6, 7, 8, 9, and 10 less 1.

L. C. M. = 2520. The number = $2520 - 1 = 2519$.

40. Let m = the man's age
 and w = the woman's age.
 $m - w$ = difference of their ages.

$w - (m - w)$ = wife's age when he was as old as she is now.

$w + (m - w)$ = her age when she is as old as he is now.

$m + (m - w)$ = his age when she is as old as he is now.

Then, $m = 2[w - (m - w)]$ (1)

and $w + (m - w) + m + (m - w) = 100$. (2)

Solving, $m = 44\frac{4}{5}$, $w = 33\frac{1}{5}$.

41. Hoover won. Curtis won too. Smith ought to win. Robinson ought to win too.

42. He charged Brown \$250 per mile for fencing, the price he evidently intended to charge Jones also. The perimeter of Jones' property being unknown, he possibly received much less.

43. By studying the relations of these numbers we find they may be arranged in pairs — 1 and 50, 2 and 49, 3 and 48, 4 and 47, 5 and 46, etc. In each case the sum of each pair is 51 and the total number of pairs is 25.

Thus $51 \times 25 = 1275$, the required sum.

44. We must first find the time between 4 and 5 o'clock when the hands are together and second find when they are opposite each other.

(1) Time past 4 when the hands are together = $20 \div \frac{11}{12}$, or $21\frac{9}{11}$ min. past 4.

(2) Time when they are opposite = $50 \div \frac{11}{12}$, or $54\frac{6}{11}$ min. past 4.

$\therefore$ time required for solving the problem = $54\frac{6}{11} - 21\frac{9}{11}$, or $32\frac{8}{11}$ min.

46. Let A , B , and C denote the husbands and a , b , and c their wives respectively. The passage was then arranged as follows:

(1) a and b cross and b returns with the boat. (2) b and c cross and c returns with the boat. (3) c remains with her husband while A and B cross. A lands while B and b return. (4) B and C cross leaving b and c . (5) a crosses with the boat and returns with b . (6) b returns for c .

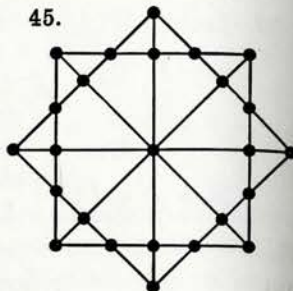
47. Each boy eats $2\frac{2}{3}$ cakes so the third boy eats $\frac{1}{3}$ of a cake out of the 3 cakes and $2\frac{1}{3}$ or $\frac{7}{3}$ cakes out of the 5 cakes. Hence he pays 1¢ to one and 7¢ to the other.

48. $9 \times 16 = 144$. $\sqrt{144} = 12$, the correct weight.

49. We must add $1\frac{1}{3}$ times the number to $\frac{2}{3}$ of it to have it doubled. Therefore $24 \div 1\frac{1}{3} = 18$, the number.

50. The number is 6×15 , or 90. The correct answer is 6×90 , or 540.

45.



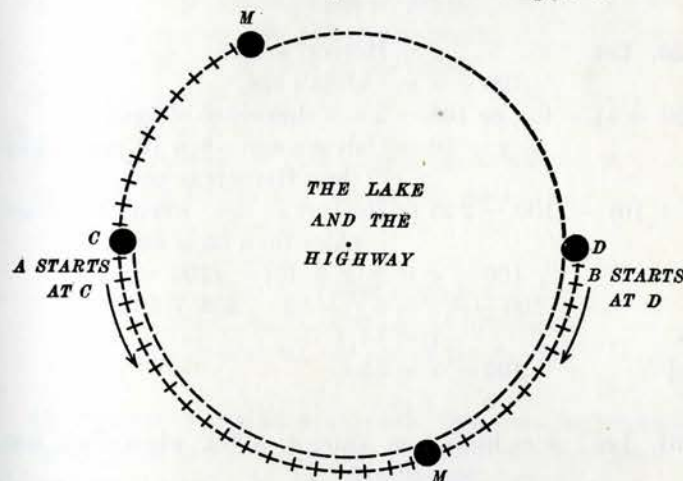
51. The stick must be $\frac{140}{100}$ of 36 inches, or 42 inches.

52. Shortest distance = $\sqrt{18^2 + (16 + 8)^2} = 30$ feet.

53. Let a = width of the track. Then he goes $2\pi(r + a)$ on the outer edge and $2\pi r$ on the inner edge. He therefore goes $2\pi a$, or 10π yards farther on the outer edge than on the inner in π seconds. In 1 second he would go $10\pi \div \pi$, or 10 yd.

54. The difference between the number and the one made by reversing its digits is always divisible by 9, because
the number = $10 \times \text{ten's digit} + 1 \times \text{unit digit}$,
its reverse = $1 \times \text{ten's digit} + 10 \times \text{unit digit}$,
and the difference = 9 times one digit minus 9 times the other digit. The difference divided by 9 gives a quotient containing as many units as there are units in the difference of the digits. Hence $27 \div 9 = 3$, the difference in the digits and the number required is 74.

55. Let $+++ \dots$ denote the path of B starting at D .
Let $--- \dots$ denote the path of A starting at C .



Also let CD = the diameter, c = circumference
and M and M' = the 1st and 2nd meeting points.

Then 421 = distance B went before meeting at M ,

$$\frac{c}{2} - 421 = \text{distance A went before meeting at } M,$$

and $\frac{c}{2} - 421 + 221$ = distance B went in going from M to M' ,

$$421 + \frac{c}{2} - 221 = \text{distance A went in going from } M \text{ to } M'.$$

$$\therefore \frac{421}{\frac{c}{2} - 421} = \frac{\frac{c}{2} - 421 + 221}{421 + \frac{c}{2} - 221},$$

(Distances traveled in equal times being proportional.)

Simplifying and collecting, we have

$$c^2 - 2084c = 0$$

or

$$c = 2084 \text{ ft.}$$

$$\therefore d = \frac{c}{\pi} = \frac{2084}{\pi}, \text{ or } 663.324 \text{ ft.}$$

56. Let x = Herbert's age.

$$100 - x = \text{Calvin's age.}$$

$(100 - x) - (x)$, or $100 - 2x$ = difference of ages.

$x + 10$ = Calvin's age when 10 years older
than Herbert is now.

$(x + 10) - (100 - 2x)$ = Herbert's age when 10 years
older than he is now.

$$\therefore 100 - x = \frac{5}{4}[(x + 10) - (100 - 2x)],$$

$$400 - 4x = 5x + 50 - 500 + 10x.$$

Or

$$x = 44\frac{1}{9}$$

and

$$100 - x = 55\frac{5}{9}.$$

57. Let x = Bill's age Mar. 4, 1909, when Taft was
inaugurated.

$3x$ = Tom's age Mar. 4, 1909, when Taft was
inaugurated.

$x + 12$ = Bill's age Mar. 4, 1921, when Harding was
inaugurated.

$3x + 12$ = Tom's age Mar. 4, 1921, when Harding was
inaugurated.

$x + 20$ = Bill's age Mar. 4, 1929, when Hoover was
inaugurated.

$3x + 30$ = Tom's age Mar. 4, 1929, when Hoover was
inaugurated.

$$\therefore 3x + 12 = 2(x + 12)$$

or

$$x = 12.$$

$$3x = 36.$$

$$x + 20 = 32, \text{ Bill's age.}$$

$$3x + 20 = 56, \text{ Tom's age.}$$

58. Let x = your age when sister was born.

$3x$ = my age when sister was born.

y = sister's age at death.

$x + y$ = your age at her death.

$3x + y$ = my age at her death.

Then, $3x + y = 2(x + y)$

(1)

or

$$x = y.$$

Also, $y + (x + y) + (3x + y) + 30 = 100.$

(2)

Substituting x for y and solving (2), we have,

$$y = 10.$$

$$\therefore x = y = 10.$$

$y = 10$, sister's age at death.

$x + y = 20$, your age at her death.

$3x + y = 40$, my age at her death.

59. Since the remainders are each 1 less than the respective
divisors, it is evident that the required number must be
1 less than the L. C. M. of the divisors, or 2519.

60. Let r = length of line.
 Area grazed over = 4840 sq. yd.
 Then, $\pi r^2 = 4800$.
 $r^2 = 4800 \div \pi = 1540.61$.
 $r = \sqrt{1540.61} = 39.25$.
 $\therefore$ length of line = 39.25 yd.

61. (a) Let x = number of shillings encompassing the circular tract

= length of circumference in inches

= value of the tract in shillings

and r = radius of the circular tract.

Then, $r = \frac{x}{2\pi}$, or $r^2 = \frac{x^2}{4\pi^2}$

and area of circle in square inches

$$= \pi \left(\frac{x^2}{4\pi^2} \right) = \frac{x^2}{4\pi}$$

Area in sq. ft. = $\frac{x^2}{4\pi} \div 144$.

Area in acres = $\left(\frac{x^2}{4\pi} \right) \left(\frac{1}{144} \cdot \frac{1}{43560} \right)$.

$$\therefore x = \frac{x^2}{4\pi} \cdot \frac{1}{144} \cdot \frac{1}{43560} \cdot 20.$$

Solving, $x = 3,941,225.16s.$, the value of the circular tract.

- (b) Let x = number of shillings encompassing the square tract

= the perimeter of square in inches

= value of the tract in shillings.

Then, $\frac{x}{4}$ = side of the square

and $\left(\frac{x}{4} \right)^2$ = area of square in square inches.

$$\left(\frac{x}{4} \right)^2 \div 144 = \text{area of square in square feet.}$$

$$\left(\frac{x}{4} \right)^2 \cdot \left(\frac{1}{144} \cdot \frac{1}{43560} \right) = \text{area in acres.}$$

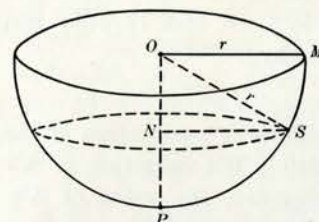
$$\therefore x = \left(\frac{x}{4} \right) \cdot \frac{1}{144} \cdot \frac{1}{43560} \cdot 20.$$

Solving, $x = 5,018,112s.$, the value of the square tract.

$\therefore$ the one who had the square tract had the greater fortune.

62. Let OM denote the radius of the bowl.

Then, $NS = \sqrt{SO^2 - NO^2} = \sqrt{12^2 - 6^2} = \sqrt{108}$.



Amount of water = volume of spherical segment of one base with radius NS and altitude PN .

Volume of spherical segment

$$= \frac{1}{2}(\pi r_1^2 + \pi r_2^2)h + \frac{\pi}{6}h^3$$

where

$$r_1 = \sqrt{108} \text{ and } r_2 = 0.$$

$$\therefore V = \frac{1}{2}(108\pi + 0)6 + \frac{1}{6}\pi 6^3$$

$$= 324\pi + 36\pi, \text{ or } 360\pi.$$

Also OM = radius of the cylinder.

Area of base of cylinder = $\pi r^2 = 144\pi$.

$\therefore$ depth of water, if in cylinder, would be $360\pi \div 144\pi$ or $2\frac{1}{2}$ inches.

A MENTAL NUT

Place three sixes together so as to make seven.

Ans. $6\frac{6}{6}$.

NUTS FOR THE MATH CLUB

1. Only a hint. What is the area of a line-triangle?
2. Beginning at one end of the chain he cuts the 5th, 14th, and 31st link. This gives him 3 cut links and 4 pieces of chain containing 4, 8, 16, and 19 links respectively. Only three cuts are necessary.
3. *1st Solution.*
B's horse plus \$20 is A's valuation of his own horse, and B's horse minus \$10 is B's valuation of A's horse; hence if they split the difference, the value of A's horse is by the agreement one-half the sum of A's and B's valuations or B's horse plus \$5; therefore B should pay A \$5.00.
- 2nd Solution.*
Since they are \$30 apart at first, each will have to throw off \$15; hence A should receive \$5.00.
- 3rd Solution.*
The difference between A and B is \$20 + \$10, or \$30. If they split the difference, A must come down \$15; consequently the boot A should receive is \$5.00.
4. Son $\frac{1}{2}$, wife $\frac{1}{4}$, and daughter $\frac{1}{4}$.
5. During the time the teacher was away from the school, the distance traveled by the minute hand added to the distance traveled by the hour hand would make the whole circle of the dial. As the minute hand moves twelve times as fast as the hour hand the spaces passed over would be $\frac{11}{12}$ and $\frac{1}{12}$ of a revolution, respectively. $\frac{11}{12}$ of a revolution of the

minute hand equals $55\frac{5}{13}$ minutes, the time the teacher was away. When the teacher left, the distance between the hands was $\frac{11}{12}$ of the distance of the minute hand from the zero point. Hence $\frac{1}{13} \times \frac{11}{12} = \frac{11}{156} = 5\frac{5}{143}$ min. past twelve, the time when he left. Add the time that he was away, $5\frac{5}{143} + 55\frac{5}{13} = 60\frac{60}{143}$, or $\frac{60}{143}$ of a min. past one, the time when he returned.

6. Let x = the number of one of the woman's hogs, and y = the number of her husband's; then by the condition of the problem $y^2 = x^2 + 63$. Consequently $x^2 + 63$ must be an integer, since $\sqrt{(x^2 + 63)}$ represents the number of hogs. The equation $y^2 - x^2 = 63$, or $(y + x)(y - x) = 63$ admits of three solutions, viz. 63×1 , 21×3 , and 9×7 .

$$\therefore \begin{cases} y + x = 63 \\ y - x = 1 \end{cases} \text{ or } \begin{cases} y + x = 21 \\ y - x = 3 \end{cases} \text{ or } \begin{cases} y + x = 9 \\ y - x = 7 \end{cases};$$

whence $y = 32$, $x = 31$; $y = 12$, $x = 9$; $y = 8$, $x = 1$. Since $32 = 9 + 23$ and $12 = 1 + 11$, 32 belongs to Hendricks, 12 to Klans, 9 to Katrine, and 1 to Gertrude.

$\therefore$ 32 belongs to Hendricks, 31 to Anna; 12 to Klans, 9 to Katrine, 8 to Hans, and 1 to Gertrude. Hence Hendricks is Anna's husband, Klans is Katrine's husband, and Hans is Gertrude's husband.

7. Let t = number of sec., n = num. of miles per hour.

$$\therefore \frac{5280 n}{3600} = \frac{22 n}{15} \text{ ft. per second} = \text{speed of train. Also}$$

in t seconds the train goes $30 n$ ft.

$$\therefore \frac{30 n}{t} = \text{number of ft. in 1 sec.}$$

$$\therefore \frac{30 n}{t} = \frac{22 n}{15} \therefore t = 20\frac{5}{11} \text{ sec.}$$

8. 200%.

9. A's land is worth $\frac{2}{3}$ as much as B's per acre; hence to divide the land so that each will receive pieces of land worth the amount, A must receive $\frac{5}{3}$ as many acres as B.

Dividing 640 acres in proportion of 5 to 3, A must receive $\frac{5}{8}$ of 640 acres, or 400 acres, and B must receive $\frac{3}{8}$ of 640 acres, or 240.

NOTE. It will be observed that the land received by each is worth only \$300. This indicates that they either paid too much for the land in the beginning, or they underestimated the value of the land when they decided it was worth \$.75 and \$1.25 per acre respectively.

10. Let x = father's age when he moved to town.

$\frac{4}{3}x$ = his age.

$\frac{1}{3}x$ = time in years since moving to town.

$\frac{1}{3}x + 3\frac{1}{3}$ = son's age in years.

$2(\frac{1}{3}x + 3\frac{1}{3} - 2)$,

or $\frac{2}{3}x + 2\frac{2}{3}$ = mother's age when they moved to town.

$x + 2\frac{2}{3}$ = mother's age.

$x - (\frac{1}{3}x + 3\frac{1}{3})$, or $\frac{2}{3}x - 3\frac{1}{3}$ = no. of years until the son's age will equal father's age (to be added to the ages of son, father, and mother).

Then, $(\frac{1}{3}x + 3\frac{1}{3}) + \frac{4}{3}x + (x + 2\frac{2}{3}) + 3(\frac{2}{3}x - 3\frac{1}{3}) = 100$.

Solving, $x = \frac{156}{7}$

$\therefore$ son's age = $\frac{1}{3}x + 3\frac{1}{3} = \frac{1}{3} \cdot \frac{156}{7} + 3\frac{1}{3}$, or $10\frac{16}{21}$ yrs.

11. My friend is given the 25- and 3-cent pieces. I am handed the 50-cent piece, 2 dimes, and the one-cent piece, and the merchant keeps the dollar, nickel, and two-cent pieces.

12. Let M , M , and M represent the three missionaries and C , C , and C' the three cannibals, C' being the one that can row the boat.

Those carried on each trip made by the boat are as follows:

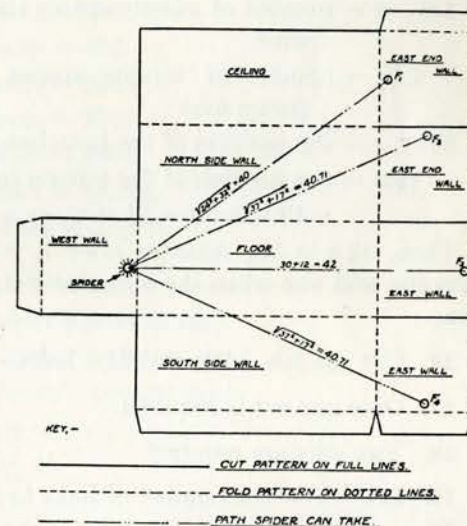
- | | | | |
|--------------|--------------|---------------|---------------|
| 1. C, C' . | 5. M, M . | 9. M, M . | 13. C', C . |
| 2. C' | 6. C, M . | 10. C' . | |
| 3. C, C' . | 7. M, C' . | 11. C', C . | |
| 4. C' | 8. M, C . | 12. C' . | |

13. It must be supposed that the rails are all of the same length. Then the three rails inclose an equilateral triangle, the area of which is three acres, or 480 sq. rd. Area of the equilateral triangle equals one-half the side squared and multiplied by the square root of 3, or by 1.73205. A side, therefore, of the triangle = 2 times the square root of the area $\div 1.73205$. The area, 480 sq. rd. $\div 1.73205 = 277.1282$ sq. rd. Two times the square root of 277.1282 sq. rd. = 33.294 rd., the length of one rail. Since the four equal rails must inclose a square, then

$33.294 \times 33.294 = 1108.49^+$, the area in square rods.

$\therefore$ the area = 6 A., 148.5 sq. rd.

14. The following pattern, taken from "School Science and Mathematics," gives a complete solution of this old but interesting problem, showing all the possible straight-line paths that the spider can take. If the figure is cut and folded as directed — printing outside — The Four F's will be found to coincide. The solutions will appear on the outside of the model.



15. Comparing this watch with a correct watch we find that this one indicates exact time while its hands are coin-

cident with those of the latter. Its minute hand is at 12 while the hour hand of the correct watch is at 6 when it is set. Since the two hands move at the same rate, they will always be 30-minute spaces apart. The same is true of its hour hand and the minute hand of the correct watch. When the two minute hands are together, the two hour hands are 30-minute spaces behind and coincident also. It is then only necessary to find the time after 6 o'clock when the hands of the correct watch are opposite. This cannot be between 6 and 7, so we find the time between 7 and 8 o'clock. This occurs once every hour, but we shall consider it only for the first case.

Let x = number of minute spaces the hour hand passes over.

$12x$ = number of minute spaces the minute hand passes over.

$35 + x$ = the position of the hour hand.

$12x$ = the position of the minute hand.

$$\therefore 12x + 30 = 35 + x, \text{ or } x = 5\frac{5}{11}.$$

Then, $12x = 5\frac{5}{11}$ minutes after 7, when the hands are opposite and also when the incorrect watch indicates correct time.

16. Cut the 5th, 14th, and 31st links.

17. Only one cut is required.

18. Two cuts are required.

19. Let n = the number of links to be cut.

Then, $n + 1$ = number of links in the shortest section of the chain.

$2(n + 1)$ = number of links in the 2nd section.

$4(n + 1)$ = number of links in the 3rd section.

$8(n + 1)$ = number of links in the 4th section.

Since there are n cut links, there must be $n + 1$ sections of the chain. Having given the first four terms of a geometrical series, we can easily find the $n + 1$, or last term, which is $(n + 1)^{2^n}$.

Finding the sum of the series and adding the n cut links, we have the maximum number of links in a chain where n cuts are required.

$\therefore$ maximum number of links = $n + (n + 1)(2^{n+1} - 1)$, where n is any positive number.

If $n = 1$, no. of links = 7.

$n = 2$, no. of links = 23.

$n = 3$, no. of links = 63.

$n = 4$, no. of links = 159.

$n = 5$, no. of links = 383.

$n = 6$, no. of links = 895.

$n = 7$, no. of links = 2047.

$n = 8$, no. of links = 4607.

$n = 9$, no. of links = 10,239.

$n = 10$, no. of links = 22,527.

20. Let x = my present age in years.

y = my sister's present age.

z = my mother's present age.

w = my father's present age.

Then, $x = \frac{1}{4}z$; $y = \frac{1}{3}w$; $y - x = \frac{1}{4}(z - x)$;

$$x + 4 = \frac{1}{4}(w + 4).$$

Solving, $x = 9.6$;

$$y = 16.8; z = 38.4; w = 50.4.$$

21. *Solution.*

(a) He added the numbers as shown to obtain the dividend.

Operation.

690 (b) He found the divisor by finding
2415 the G. C. D. of 690, 2415, and 2070, which
2070 is 345.
1 (c) He then reproduced the solution as
95221 follows:

$$\begin{array}{r} 276 \\ 345 \overline{) 95221} \\ \underline{690} \\ 2622 \\ \underline{2415} \\ 2071 \\ \underline{2070} \\ 1 \end{array}$$

22. Largest . . £98,765 4s. $3\frac{1}{2}d.$
Smallest . . 2,567 18s. $9\frac{3}{4}d.$

23. Since there were 5 pastures containing an equal number of animals, the number must be a multiple of 5. Also since the eight dealers bought an equal number of animals, the number must be a multiple of 8. Therefore the number is a multiple of 40. The highest multiple of 40 that will work is 120. This number of animals could be arranged in one of two ways — 1 cow, 23 hogs, and 96 sheep, or 3 cows, 8 hogs, and 109 sheep. But since the animals consist of cows, pigs, and sheep, the first must be excluded. The correct number is therefore 3 cows, 8 hogs, and 96 sheep.

24. (a) If the second remainder is less than the first, take the difference of remainders. (b) If the second remainder is greater than the first, take the difference of remainders from 9. (c) If remainders are equal, the digit erased was either 9, or 0.

Ex. Suppose 732 is selected.

Dividing by 9, the remainder is 3; erasing 3 and dividing, the remainder is 0. Hence the number erased is $3 - 0$, or 3.

Ex. Suppose 523 is selected.

Dividing by 9, the remainder is 1; erasing 5 and dividing 23 by 9 gives a remainder of 5. Hence the number erased is $9 - (5 - 1)$, or 5.

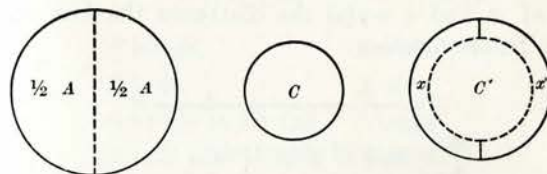
25. 3025.

26. Only one. The fish can be divided into any two fractional parts such as $\frac{1}{3}$ and $\frac{2}{3}$, provided the denominator equals the sum of the two numerators.

27. Since A was to furnish half of the feed, B owes him 30 ears. If A should be given 30 ears from the load, he would be receiving only 20 ears from B since he owns a third interest. $\therefore$ 45 ears should be given A.

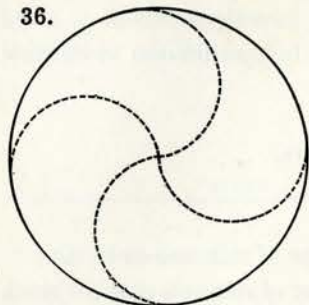
28. I could not see 36 dozen and could see 12 dozen. I therefore had 48 dozen.

29. Since $\pi a^2 = \pi b^2 + \pi c^2$, it follows that the two smaller pies are exactly equal to the larger pie. Then if A, B, and C represent the pies, the division is made as follows:

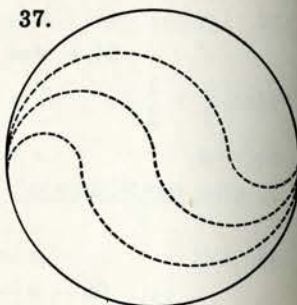


Two of them receive $\frac{1}{2} A$ each. The shares of the other two are found by placing C upon B and tracing its circumference as shown. One receives C and x, while the other receives C' and x'.

36.

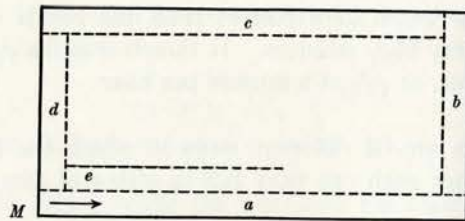


37.



38. A brick weighs 3 lb. Then 16 brick weigh 48 lb. and 11 brick weigh 33 lb. The boy's weight equals $\sqrt{48 \times 33}$, or 39.79 lb.

39. *Solution I.* The diagram below represents the field. The farmer starts his machine at M , cuts the strip a , down the entire length of the field, cuts b , then c , d , and is then



ready to repeat, starting on strip e . Although the lengths a , b , c , and d are all of different lengths, he has cut an area which is equal to the difference in area of two rectangles. The original area is $l \cdot w$; the part remaining after the first round is $(l - 2x)(w - 2x)$ where x is the width of a swath. After 11 rounds the area remaining is $(l - 22x)(w - 22x)$.

If k be the width of a swath,
then, $28k = w - 22k$, or $w = 50k$.

Also $\frac{lw}{2} = (l - 22k)(w - 22k)$, or $3l = 616k$.

$$\therefore \frac{l}{w} = \frac{308}{75}$$

Solution II. Using one swath as a unit, the width of the field = 50 swaths.

Let l represent the number of swaths in the length of the field.

Then, $28(l - 22) = 25l$, or $l = \frac{616}{3}$.

$$\therefore l:w = \frac{616}{3}:50, \text{ or } \frac{308}{75}.$$

40. It is evident that A has 2 more games to win while B has 4, and excluding the possibility of tie games, at most 5 games only will be necessary. Tie games need not be considered since they neither affect the existing standing nor change the number of games remaining to be played. As the probability of either winning is equal, $p = q = \frac{1}{2}$. Then if we let $r = 2$, the games A must win, and $n = 5$, the greatest number of decisive games to be played, the conditions are those of Pascal's probability problem solved by the formula

$${}_nC_r p^r q^{n-r} + {}_nC_{r+1} p^{r+1} q^{n-r-1} + \dots + {}_nC_{n-1} p^{n-1} q + p^n$$

which for this case becomes

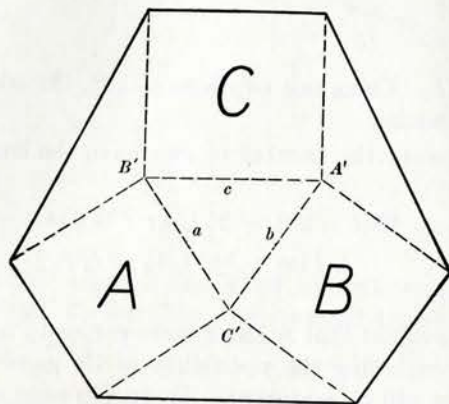
$$\frac{1}{32} + \frac{1}{32} + \frac{5}{32} + \frac{1}{32} = \frac{13}{64}.$$

Since the probability of A's winning is $\frac{13}{64}$, the odds in favor of A are 13 to 3. Bernoulli's formula ($n = 2$, $m = 4$, $p = q = \frac{1}{2}$) gives the same result.

— From "School Science and Mathematics."

41. When John takes 12, James is entitled to 9 and Sam to 14. John is then entitled to $\frac{12}{35}$ of the marbles, or 264. James to $\frac{9}{35}$, or 198, and Sam to $\frac{14}{35}$, or 308.

42. Let a , b , and c be the sides of the central $\triangle$ and A' , B' , and C' the vertices. Then its area $= \frac{1}{2} ab \sin C' = \frac{1}{4} ab \sqrt{2}$,



since $\sin 45^\circ = \frac{1}{2}\sqrt{2}$. The area of each of the three outer $\triangle$ equals the area of $\triangle A'B'C'$ since supplementary angles have the same sine.

$$\begin{aligned}\text{Also } c &= \sqrt{a^2 + b^2 - 2ab \cos C'} \\ &= \sqrt{a^2 + b^2 - ab\sqrt{2}}.\end{aligned}$$

Then area of 4 $\triangle = ab\sqrt{2}$.

$$\text{area of } A = a^2.$$

$$\text{area of } B = b^2.$$

$$\text{area of } C = a^2 + b^2 - ab\sqrt{2}.$$

$$\therefore \text{entire area} = 2(a^2 + b^2) = 2(4.73 + 5.27), \text{ or } 20 \text{ acres.}$$

43. In Geometry we get the following formula:

$$\text{area of } \triangle = \frac{1}{4}\sqrt{4a^2b^2 - (a^2 + b^2 - c^2)^2}.$$

Changing values of a , b , and c from the sides of a $\triangle$ to areas of the squares, the formula becomes

$$\text{area of } \triangle = \frac{1}{4}\sqrt{4ab - (a + b - c)^2}.$$

Since $a = 18$, $b = 20$, and $c = 26$, by substituting, we find the area of the central $\triangle$ to be 9 acres

$$\text{area of 4 } \triangle = 36 \text{ acres}$$

$$\text{area of } A = 18 \text{ acres}$$

$$\text{area of } B = 20 \text{ acres}$$

$$\text{area of } C = 26 \text{ acres}$$

$$\therefore \text{entire area} = 100 \text{ acres}$$

44. Six cuts are required to make a straight chain while 7 cuts are required to make an endless chain. The answers are therefore 24 cents and 28 cents.

45. There is supposed to be no other such number.

46. Let $\left(\frac{y}{x}\right)$ be the square proper fraction.

$$1 - \left(\frac{y}{x}\right)^2 = \left(\frac{n}{m}\right)^2, \text{ or } m^2(x^2 - y^2) = n^2x^2.$$

Let $x = p^2 + q^2$, and $y = p^2 - q^2$.

$$\therefore 1 - \left(\frac{p^2 - q^2}{p^2 + q^2}\right)^2 = \frac{4p^2q^2}{(p^2 + q^2)^2}, \text{ where } p \text{ can have any value greater than } q.$$

Let $p = 2$ and $q = 1$.

$$\therefore 1 - \left(\frac{3}{5}\right)^2 = \left(\frac{4}{5}\right)^2.$$

Let $p = 3$, $q = 2$.

$$\therefore 1 - \left(\frac{5}{13}\right)^2 = \left(\frac{12}{13}\right)^2.$$

47. Since the rock weighed 40 pounds, the weights of the pieces are 1, 3, 9, and 27 pounds respectively.

48. Let a , b , and c = the respective number of eggs the three women sold at y cents for every d eggs; and for the remaining eggs let x cents = price per egg.

Then,

$$(10 - a)x + \frac{a}{d}(y) = (30 - b)x + \frac{b}{d}y = (50 - c)x + \frac{c}{d}y.$$

Whence

$$y = \left(\frac{b-a-20}{b-a} \right) dx = \left(\frac{c-a-40}{c-a} \right) dx = \left(\frac{c-b-20}{c-b} \right) dx.$$

Solving for a , b , and c , we find $a + c = 2b$.

For positive values,

$$a < 9, b > 20 + a \text{ and } < 31, c > 40 + a \text{ and } < 51.$$

Put $d = 2$. Take $a = 2$. Then $b = 24$ and 26 , $c = 46$ and 50 , and $y = \frac{2x}{11}$ and $\frac{1}{3}x$.

For integral values, put $x = 11$ and 3 respectively.

Then, $y = 2$ and 1 .

$\therefore$ 10 eggs brought $\frac{2}{3} \times 2c + 8 \times 11c = 90c$, or $\frac{2}{3} \times 1c + 8 \times 3c = 25c$; 30 eggs brought $\frac{2}{3} \times 2c + 6 \times 11c = 90c$, or $\frac{2}{3} \times 1c + 4 \times 3c = 25c$; and 50 eggs brought $\frac{2}{3} \times 2c + 4 \times 11c = 90c$, or $\frac{2}{3} \times 1c = 25c$.

Put $d = 3$. Take $a = 3$ and 6 . Then $b = 24$ and 27 , $c = 45$ and 48 , and $y = \frac{1}{3}x$.

Put $x = 7$, then $y = 1$, and each of the women received 50¢ or 30¢.

49. Let x = the number of apples in the bag.

a, b, c = the number taken at night by A, B , and C , respectively.

s = share of each next morning.

Then,

$$x - 1 = 3a; x - 2 - a = 3b; x - 3 - a - b = 3c,$$

$$\text{and } x - 4 - a - b - c = 3s.$$

$$\therefore 8x - 81s = 65.$$

Solving for integral values of x , we find that x equals each term of the series, 79, 160, 241, ... $(79 + 81n)$.

50. The yield being an arithmetical mean, the rent should likewise be one.

$$\therefore \frac{1}{2} \left(\frac{1}{2} + \frac{2}{3} \right), \text{ or } \frac{9}{20} \text{ of } 60 \text{ bu.} = 27 \text{ bu.}$$

51.

$$.000\frac{1}{2} = \frac{1}{2} \text{ of } .001.$$

$$.00\frac{1}{2} = \frac{1}{2} \text{ of } .01.$$

$$.0\frac{1}{2} = \frac{1}{2} \text{ of } .1.$$

$$\therefore \frac{1}{2} = \frac{1}{2} \text{ of } 1, \text{ or } \frac{1}{2}.$$

Then,

$$2^2 = 4.00$$

$$2^{-2} = \frac{1}{4} = .25$$

$$2^{2-2} = 2^0 = 1.00$$

$$.2^{-2} = \frac{1}{4} = .25$$

$$\frac{1}{.2} = 5.00$$

$$\frac{1}{2} = .50$$

$$\frac{1}{2} = \frac{1}{20} = .05$$

$\therefore$ the dumb man had \$11.05

52. When a quadrilateral is concyclic, its area may be found by using the following formula (see 21, page 152).

area = $\sqrt{s(s-a)(s-b)(s-c)(s-d)}$, where a, b, c , and

d denote the sides and $s = \frac{1}{2}(a+b+c+d)$.

area = 30,397.31 sq. rd.

53. *Solution.* Let xy and zy represent the rivers and A and B the places where A and B live.

Draw $AM \perp$ to xy and produce it to A' , making $MA' = MA$.

Draw $BN \perp$ to yz and produce it to B' , making $NB' = NB$.

Draw $B'A'$ cutting xy at D and yz at D' .

Then $ADD'B$ is the shortest path (being equal to $A'B'$).

54. Obviously the money cannot be divided strictly into these fractional parts because their sum is not unity. Therefore it must be divided in the ratio of $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{1}{6}$. The sum of these fractions is $\frac{37}{60}$. The first should receive $\frac{37}{60}$, the second $\frac{15}{60}$, the third $\frac{12}{60}$, and the fourth $\frac{10}{60}$ of \$1.

55. Assuming the person to be born in the last century, the year of birth may be represented by $1800 + 10y + x$; t years later we may call it $1800 + 10u + z$, and another t years later by $1800 + 10w + v$. We get five equations with seven unknowns as follows: $9 + x + y = 9 + u + z = 9 + w + v = 10u + z - 10y - x = 10w + v - 10u - z = t$. From these we find $x + v = 2z$; $y = 8x - 9z - 9$; $w = x + y - v$; $u = w + v - z$; $t = w + v + 9$.

Since all unknowns must be positive and less than 10, by solving we find $t = 18$ and the years of birth of the person are 1809, 1818, 1827, 1836, 1844, 1872, 1890, 1908, and 1917, which correspond to an age at the present time (1931) of 122, 113, 104, 95, 86, 79, 59, 41, 23, 14.

58. Since C dug as many feet as A and B combined, he dug $\frac{1}{2}$ of 240 rods or 120 rods, the number of rods of rock. Since he could have dug 240 rods of rock in 34 days, he must have dug the 120 rods of rock in 17 days, the time required for the work to be completed. A and B must then have dug 120 rods of clay and sand in 17 days.

A can dig 4 rods of sand or 3 rods of clay in a day, while B can dig 5 rods of sand or 2 rods of clay in a day. It is evident that there are more rods of clay than sand, because at the rates of 3 rods of clay and 5 rods of sand per day the 120 rods would have been completed in 15 days, provided that B dug in sand all the time. In 17 days A could dig 51 rods of clay and B 85 rods of sand, making 16 rods more than required. Hence B must have worked part of his time in clay.

Now since he would dig 3 rods less per day in clay than in sand, to dig 16 rods less he must have worked $16 \div 3$, or $5\frac{1}{3}$ days in clay, and $17 - 5\frac{1}{3}$, or $11\frac{2}{3}$ days in sand. Hence there are

$11\frac{2}{3} \times 5$ rd. = $58\frac{1}{3}$ rd., the number of rods of sand,
 120 rd. - $58\frac{1}{3}$ rd. = $61\frac{2}{3}$ rd., the number of rods of clay,
 and 120 rods of rock.

59. Let x = the number selected.

Then performing the successive operations except (g), we have the complex fraction

$$\frac{1}{1 - \frac{1}{1 - \frac{1}{1 - x}}},$$

which when simplified reduces to x , the number selected.

To reveal the original number selected, therefore, the proposer has only to divide the final result by the multiplier used in (g).

The work is not complicated if the result of each operation is simplified before the next operation is performed.

Thus the steps may be written

(a) $1 - x.$

(d) $\frac{x - 1}{x}.$

(b) $\frac{1}{1 - x}.$

(e) $1 - \frac{x - 1}{x} = \frac{1}{x}.$

(c) $1 - \frac{1}{1 - x} = \frac{-x}{1 - x}.$

(f) $\frac{x}{1} = x.$

(g) $2x.$

$\therefore$ number selected = $2x \div 2 = x.$

60. The score of 100 was made in six shots, two of which were upon the 16 and four upon the 17.

61. (a) Nineteen steps required. First mount step 1, then go back to the ground, and continue step by step thus: 1, 2, 3, 2, 3, 4, 5, 4, 5, 6, 7, 6, 7, 8, 9, 8, 9.

(b) Twenty-three steps required. First step to the first rung, then back to the ground; then to 1 and 2, back to 1, up to 2 and 3, etc., always one step down and two up, until the top rung is reached.

62. The dividend always equals the product of the divisor and quotient. If in this case we should multiply the divisor (once the number *plus* 4) by the numerator (which is 3) of the quotient, we get 4 times the number, because once the number is only $\frac{1}{4}$ of 3 times, or $\frac{3}{4}$ of the divisor. Then 3 times (the number *plus* 4) equals 4 times the number, or 3 times the number *plus* 12 equals 4 times the number, and the number is 12.

63. Let A , B , and C be the three digits, reading from left to right. Then

$$A 7^2 + B 7 + C = C 9^2 + B 9 + A.$$

Hence $48A = 2B + 80C$, or $3A = \frac{B}{8} + 5C$.

Now A , B , and C are integers; then $3A$ is a whole number.

Hence $\frac{B}{8} + 5C$ must be a whole number. For this to be true, B must be 0, or a multiple of 8. But 8 does not exist in the scale of 7, and B must be 0. Then $3A = 5C$. The smallest integers that satisfy this equation are $A = 5$ and $C = 3$. Hence the digits are 5, 0, and 3.

In the scale of 10 the number is 248.

A BRAIN NUT

If 6 dozen eggs cost as many cents as I can buy eggs for 50 cents, what is the price of eggs per dozen?

Solution. Let x = the cost per dozen in cents.

Then $6x$ = the cost of 6 dozen.

Also $\frac{50}{x}$ = number of eggs bought for 50 cents.

$$\therefore 6x = 12 \left(\frac{50}{x} \right)$$

$$\text{or } x^2 = 100.$$

$$\therefore x = 10, \text{ the cost in cents per dozen.}$$

NUTS FOR THE MAGICIAN

1. To the number eleven.

4. In every case the sum of the digits in the result is 9. If I am given 5, I simply say $9 - 5 = 4$, the other digit.

NOTE. The reason for this is that both numbers having the same digits are multiples of 9 with the same remainder or excess. Their difference is therefore an exact multiple of 9, and consequently the sum of the two digits is 9. When the digits of the number are equal, the difference is 0, and when they differ by 1, the difference is 9.

5. The quotient always consists of two equal digits. This digit is found by taking the difference between the first and last digit of the number taken.

Let a , b , c denote the hundreds', tens', and units' digits, respectively, of any number of three digits.

Then, $\text{Number} = 100a + 10b + c$

$\text{Number (rev. digits)} = 100c + 10b + a$

$\text{Difference} = 99a - 99c = 99(a - c)$

$99(a - c) \div 9 = 11(a - c)$, the quotient.

Since the quotient consists of the product of 11 and a number consisting of a single digit, therefore both figures are alike and may be read backward or forward.

6. The figure erased = 1st rem. - 2nd rem.; or if the 1st is not greater than the second.

The figure erased = (1st rem. + 9) - 2nd rem.

NOTE. The reason for this is that the remainder, after dividing a number by 9, is the same as the remainder after dividing the sum of the digits by 9, and hence the sum of the digits being diminished by the number erased, the remainder will also be diminished by it. If there is no remainder either time, then the figure erased was either 0, or 9.

7. Suppose 45x493 is the number written. Write 2 in the place of x . It is the excess found by casting out the 9's, or the difference between the sum of the digits and the higher multiple of 9.

NOTE. This may be varied by filling in the space with a figure that will make the number divisible by 9 and leaving a specific remainder.

8. The canceled digit is found by casting the 9's out of 21, giving an excess of 3, and subtracting from 9, leaving 6, the canceled digit.

The canceled digit may also be found by subtracting the sum of the digits given from the higher multiple of 9. Thus, $27 - 21 = 6$.

9. The canceled digit is found identically as was done in Ex. 8.

10. In obtaining the remainder I simply multiply the first remainder by the number which I named as a multiplier.

Ex. Let 32061 be the number written and 11 the multiplier named.

Casting the 9's out of 132061, we have a remainder of 4.

Casting the 9's out of 11×132061 , we have a remainder of 8.

The last remainder 8 is found by casting the 9's out of 11×4 , or 44.

11. The work being correct the middle figure will always be 9. The first figure is found by subtracting the last one from 9, which is the sum of the first and last figures.

12. The answer for any number taken is 1089.

Explanation. Let a, b, c denote the three digits. Then we have

$$\begin{aligned} 100a + 10b + c &= n \\ 100c + 10b + a &= n_1 \\ 100(a - c) + (c - a) & \end{aligned}$$

$$\begin{array}{l} \text{or} \quad 100(a - c - 1) + 90 + (10 + c - a) = d \\ \text{and rev. dig. } 100(10 + c - a) + 90 + (a - c - 1) = n_2 \\ \text{Adding,} \quad \frac{900}{\quad} + \frac{180}{\quad} + \frac{9}{\quad} = 1089 \end{array}$$

13. The answer will always be 198. All the pupils will have the same result although each chose his own number.

14. The sum of the first and last digits is 10 while the sum of the middle two is 8. The first figure is found by subtracting the last from 10, while the digit in hundreds' place is found by subtracting the one in tens' place from 8.

15. The result is always 0. This problem is similar to Ex. 12. Here we have a number of four figures instead of three.

In every case after reversing the order of the digits and adding, the result will be 10,890. This may be explained as follows:

Let A, B, C, D denote the digits where $A > B > C > D$. Write the number in our decimal system, using the letters just as we use digits, and "borrow" and "carry" as in our decimal arithmetic. Brackets are used only to separate digits.

Given number

$$= [A] \quad [B] \quad [C] \quad [D]$$

Number rev. digits

$$= [D] \quad [C] \quad [B] \quad [A]$$

Difference

$$= [A - D] \quad [B - 1 - C] \quad [9 + C - B] \quad [10 + D - A]$$

Rev. digits in dif.

$$= [10 - A + D] \quad [-B + 9 + C] \quad [-1 - C + B] \quad [-D + A]$$

$$\therefore \text{sum} = \frac{10}{\quad} \quad \frac{8}{\quad} \quad \frac{9}{\quad} \quad \frac{0}{\quad}$$

Subtracting 889 leaves the symmetrical number 10001.

Reversing the digits and subtracting, the difference is found to be 0.

For example, suppose the number 8531 is written. The steps taken would then be as follows:

$$\begin{array}{r}
 \text{Given number} = 8531 \\
 \text{Number rev. digits} = 1358 \\
 \text{Difference} = \underline{7173} \\
 \text{Rev. digits in dif.} = \underline{3717} \\
 \hline
 10890 \\
 \text{Subtracting} \quad 889 \\
 \text{Difference} = \underline{10001} \\
 \text{Rev. digits in dif.} = \underline{10001} \\
 \hline
 \text{Subtracting, the result} = 0
 \end{array}$$

16. The number is found by subtracting 1 from the difference between the squares and dividing by 2.

Solution. Let x = the number thought of by your friend.

Then, $x + 1$ = next larger number

and $(x + 1)^2 - x^2$ = difference between the squares.

Simplifying,

$$x^2 + 2x + 1 - x^2 = 2x + 1, \text{ the difference given you.}$$

You then find the number by subtracting 1 and dividing by 2.

17. Subtract 111 from the result and you have the two numbers.

Let a and b denote the two digits in the order written.

Then the number formed is $10a + b$.

We then have

$$(2a + 1)5 + b + 106 - 111 = 10a + b, \text{ the number.}$$

18. I only subtract 250 from the number given me and I have the three digits in their order as written.

Let a, b, c denote the three digits in the order written.

Then we have

$$[(2a + 5)5 + b]10 + c - 250,$$

or, simplifying, we have,

$$100a + 10b + c, \text{ the number represented by them in order.}$$

19. I subtract 555 from the result given me and have the four numbers in their order.

Let a, b, c, d denote the four digits. Then the number formed by them is,

$$[([(2a + 1)5 + b]2 + 1)5 + c]2 + 1]5 + d - 555,$$

or, simplifying, we have

$$1000a + 100b + 10c + d, \text{ the number.}$$

20. This may be explained as follows:

Let a = your age

and b = age of an older person.

The successive operations are then

$$99 - a + b - 100 + 1 = b - a.$$

21. If the number consists of

2 digits, the difference is	9
3 digits, the difference is	198
4 digits, the difference is	3,087
5 digits, the difference is	41,976
6 digits, the difference is	530,865
7 digits, the difference is	6,419,754
8 digits, the difference is	75,308,643
9 digits, the difference is	864,197,532
10 digits, the difference is	9,753,086,421

22. Each time I wrote a number making each corresponding figure a complement of 9. I did not, however, consider your numbers in the order given. I took them in reverse order, considering the last one first. Again, to make the problem more mystifying to you I read the numbers backward. I thus read 253 backward as 352 and wrote 647. I then read 842 backward as 248 and wrote 751.

To reveal the canceled figure I simply cast the nines out of the number (20) given me and subtracted the remainder (2) from 9, giving 7, the canceled digit. If the remainder is 0, the number canceled is 9.

23. The number 39,996 is 4×9999 . If each time the professor wrote a number making each corresponding digit a complement of 9, it is evident that the sum of the eight numbers would be equal to the number shown them. He first read the number 4537, writing the complement of each digit, and thus wrote the number 5462. He then read 6314 and wrote 3685. He then read 2108 and wrote 7891. Finally he read 5364 and wrote 4635.

24. Since the sum of the figures in each column, row, and diagonal is 27, the missing figure can easily be found by adding the remaining figures and subtracting from 27. In cases (4) and (5) simultaneous equations may be required.

25. The missing figure in each case is found by subtracting the sum of the digits given you from the higher multiple of 9.

Many such groups of numbers may be written. In the following group the sum of the figures in each row, column, and diagonal is 36.

8	6	6	6	3	7
6	4	7	6	5	8
4	6	5	7	8	6
7	8	4	6	6	5
5	7	6	7	7	4
6	5	8	4	7	6

26. (1) The little girl added the first column, getting 13, and took it from 18, the higher multiple of 9, leaving 5, the first digit in her number. She then added the second column, saying, mentally, 1, 3, 7. She then subtracted 7 from 9 and wrote 2, the second digit in her number. Then she added the third column, calling results 2, 9, 10, or simpler 2, 0, 1. (She dropped the 9 so as to have a smaller number to carry in her mind.) She then said $9 - 1 = 8$, the third digit in her number. Adding the last column, she said 6, 10, 1, 4. Finally she said $9 - 4 = 5$, the last digit to be found.

The professor now canceled the 7 and added the other figures, obtaining the sum 19208. He then found the sum of the digits to be 20, which he gave to the little girl.

She then revealed the canceled figure by taking 20 from the next higher multiple of 9.

(2) She first added the numbers at the top 1, 2, 4, saying 1, 3, 7. Then $9 - 7 = 2$, the first figure at the top of her column. Then she said 4, 0, 3 and $9 - 3 = 6$, the second figure. Likewise the other figures are found to be 8, 2, 8, and 0.

The instructor then cancels the 5 and adds as before. He then gives 31, the sum of the digits, to the little girl.

She finds the canceled figure by saying $36 - 31 = 5$, or by saying $9 - (3 + 1) = 5$.

(3) The little girl found the numbers 1, 6, 5, 6, 2 as in (1) and (2) and wrote the figures 5, 6, and 9 by adding 1, 6, and 2. Had she written only two figures, 5 and 6, and permitted the 9 to vanish, the result would not have been altered.

The professor canceled the 6, added, and gave 21, the sum of the digits, to the little girl.

She revealed the canceled figure by mentally saying $2 + 1 = 3$ and $9 - 3 = 6$.

(4) Yes, she was right. In either (1), (2), or (3) she could have written a single figure which would have answered her purpose just as well. For example, in (1) instead of writing 5825 she could have written only the figure 2, found by adding the figures written by the professor as shown in (1) and (2). The figure 2 may also be found by casting the 9's out of 5825, the number written by her.

27. The blocks which the boy stacked bore the numbers 1, 2, 3, 6, 7, and 8, the sum of which is 27, a multiple of 9. Had the last block been in the stack the sum would have been a multiple of 9. The father therefore subtracted the

sum of the digits of the sum given him by his son from the next higher multiple of nine. The blocks bearing the numbers 4 and 5 were not needed since their sum was a multiple of 9.

28. I first find the sum of the digits:

$$1 + 3 + 1 + 3 + 9 = 17.$$

I find the digit not used by subtracting 17, the sum of the digits, from the next higher multiple of nine.

29. (a) Cast the nines out of the sum. The remainder left over will be the digit which was added to 9.

(b) Subtract the sum of the digits from the next higher multiple of 9.

30. In each case she subtracted from 9 or the next higher multiple of 9.

31. She did as in Ex. 30.

34. Subtract the sum of the digits of the sum from the next higher multiple of 9.

Thus, $6 + 1 + 0 + 1 + 4 = 12$
and $18 - 12 = 6$, the digit required.

35. The missing digit is found as in Ex. 34.

36. The multiplier is found by multiplying the selected number by 9. Thus if 3 is selected, the multiplier is 27. The result for any number selected is given below.

$$\begin{aligned} 12345679 \times (1 \times 9) &= 111,111,111. \\ 12345679 \times (2 \times 9) &= 222,222,222. \\ 12345679 \times (3 \times 9) &= 333,333,333. \\ 12345679 \times (4 \times 9) &= 444,444,444. \\ 12345679 \times (5 \times 9) &= 555,555,555. \\ 12345679 \times (6 \times 9) &= 666,666,666. \\ 12345679 \times (7 \times 9) &= 777,777,777. \\ 12345679 \times (8 \times 9) &= 888,888,888. \\ 12345679 \times (9 \times 9) &= 999,999,999. \end{aligned}$$

The explanation is simple, the number 12345679 being the repetend of the fraction $\frac{1}{81}$ and $9 \times \frac{1}{81} = \frac{1}{9}$, whose repetend is .111,111 ..., while 27 is 3×9 . Thus 9 times the number chosen is the multiplier.

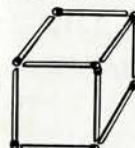
38. In each case the total should be twice the current year.

39. (a) Form a tetrahedron as illustrated.



(b) If the matches are to lie in the same plane, the best

solution is



40. (a) $V I + I V = X$

(b) $M V - T = V$

(c) $V - I V = V T$

41. (a) $st + e = \frac{1}{2}$ of st(rik)e.
 $on = \frac{2}{3}$ of t(on).

$\therefore st + on + e = \text{stone}.$

(b) L + O + V + E, the E being one of the five letters in the word "eight."

(c) Write the words "twelve" and "seven" thus,
TW(EL)VE and S(EVEN).

Then "el" is one third of twelve and "even" is four fifths of seven.

Then adding $EL + EVEN = ELEVEN.$

42. Suppose the person's date of birth is 1880 and the current year 1936. The person would then write as follows:

Date of birth =	1880
Add 4	4
	1884
Adding age (56)	56
	1940
Subtracting (31 for year 1936),	31
Multiplying by 1000	1,909,000
Subtracting 694423	694423
	1214577

Substituting letters, we have ABADEGG.

43. April Fool.

44. (a) In answer to the first question, they stepped into the order 316.

(b) The arrangement was 136.

(c) After trying all the arrangements of the three numbers, the girl who wore the number "six" took her place between the other two and stood on her head so as to form the number 391.

(d) They arranged themselves in the order 631 and the girl wearing 1 stepped on a chair, forming the number 63¹ expressing 63.

(e) The girl wearing 6 held the girls wearing 1 and 3 so as to form the fraction $\frac{3}{6}$, expressing $\frac{1}{2}$. Similarly the girl wearing 3 held the girls wearing 6 and 1 so as to form the fraction $\frac{6}{3}$, expressing 2.

(f) They arranged themselves in the order 316 and the girl wearing 6 stood on her head on a chair, forming the number 31⁶, thus forming the largest number possible.

46. From the result subtract 1111. The figures on the left will be the person's year of birth and the figure, or figures, on the right the person's age. Thus, if the result given you is 193,131 subtract 1111, giving 192,020; then state that the person's year of birth is 1920 and his age is 20.

GINGERBREAD NUTS

This is propounded in the shape of a conjuring-trick, usually after two or three *bona-fide* tricks have been performed. You place three gingerbread nuts* on the table, and cover each with a borrowed hat. You make a great point of having nothing concealed in your hands, and profess your willingness to allow the audience, if they please, to mark the three articles, so that there can be no question of substitution.

You then take up each hat in succession, pick up the nut beneath it, and gravely eat it, replacing the hat mouth downward on the table. Any one is at liberty to see that there is nothing left under either hat. You then undertake to bring the three nuts under whichever of the three hats the company may select; and the choice being made, you at once do so.

How is it to be done?

This is a very ancient "sell," but it still finds victims. The performer's undertaking is performed by simply putting on the hat selected. No one can deny that the ~~three~~ ^{one} nuts are thereby brought under the hat.

* In default of gingerbread nuts, three almonds, raisins, or any other small eatable articles may be substituted.

NUTS FOR THE PROFESSOR

a 1. For the train to pass completely through the station it must run a distance equal to the length of the station plus its own length. For the train to pass the man in 10 seconds it must run a distance equal to its own length in that time. Therefore 24 seconds - 10 seconds, or 14 seconds, is the time required for the train to run 308 yards. $\frac{1}{14}$ of 308, or 22 yards, is the distance the train runs in one second.

$\therefore$ length of train = 10×22 yd., or 220 yards.

and rate of train in miles per hour = $\frac{60 \times 60}{1760} \times 22$, or 45 miles.

a 2. Any number equals

10 times tens' digit + 1 times units' digit.

Its reverse is

1 times tens' digit + 10 times units' digit.

The difference is always 9 times one digit minus 9 times the other, and thus is divisible by 9.

Also 9 is contained as many times in the difference as there are units in the difference between the units. Thus $83 - 38 = 45$, which is 5×9 , and 5 is the difference between 8 and 3.

a 3. To pass going in opposite directions, their combined speed is $(500 + 300) \div 10$, or 80 feet per second, and when moving in the same direction, their difference of speed is $(500 - 300) \div 25$, or 8 feet per second. The swifter train, therefore, moves $\frac{1}{2}$ of $(80 + 8)$ or 44 feet per second, or 30 miles per hour, while the slower train moves $\frac{1}{2}$ of $(80 - 8)$ or 36 feet per second, or $24\frac{6}{11}$ miles per hour.

a 4. The 12th power of a number divided by its 11th power gives the number itself, and the only number whose square root equals its cube root is 1.

$\therefore 89,798 \times 1 = 89,798$.

a 5. Suppose a 4-foot section of the surface of the cylinder unrolled and forming a plane surface. Then the diagonal of this rectangle (4 feet long and 3 feet wide) will be 5 feet the length of the tape once round. The length of the tape must therefore be $10\sqrt{(4^2 + 3^2)}$, or 50 feet.

a 6. The autos meet in one hour; therefore if the bee is flying all the time, it will fly for one hour, or 15 miles.

This may also be solved by progression as follows:

The time taken by the bee for the first trip is $\frac{4}{5}$ hour.

Since $15x + 10x = 20$

$x = \frac{4}{5}$ hour

The time for second trip is

$15x + 10x = 4$

$x = \frac{4}{25}$ hour

or for successive trips,

$\frac{4}{5}, \frac{4}{25}, \frac{4}{125}, \frac{4}{625} \dots$ hours

The sum must be one hour.

These four terms add to $\frac{924}{625}$.

So add $\frac{1}{625}$ hour to make one full hour.

The distance for each trip will be the sum of these, multiplied by 15.

$\frac{60}{5}, \frac{60}{25}, \frac{60}{125}, \frac{60}{625}, \frac{15}{625}$ miles

Sum = $\frac{9375}{625}$ miles or 15 miles.

The bee at this rate will be in mid-air when the two cars meet, hence we should add $\frac{4}{3125}$ hour where we have added $\frac{1}{625}$ hour, to allow the bee to land on the next radiator.

a 7. If 5 apples are worth 2 cents, 20 apples or 25 plums are worth 8 cents; 35 apples or 15 pears are worth 14 cents, which is the cost of 8 apricots, and 4 apricots cost 7 cents.

Then the L. C. M. of 5, 25, 15, and $4 = 300$, the least number of each kind that can be taken.

Their cost = $\$1.20 + \$.96 + \$2.80 + \5.25 , or $\$10.21$.

a 8. C will first collect his claim of \$5000. B then gets his \$2500, and A must be satisfied with \$7500, thus losing \$2500 and his attorney's fees.

a 9. Since 160% represents the selling price for the correct measure, and 140% for the incorrect measure,

$$\therefore 140 : 160 = 36 : x.$$

Solving, $x = 41\frac{1}{2}$ inches, the length of the measure used.

a 10. Since the dimensions of the larger are $1\frac{1}{2}$ times those of the smaller, the area of the larger is the square of $1\frac{1}{2}$, or $2\frac{1}{4}$ times as much, and should cost $2\frac{1}{4}$ times \$800, or \$1800.

a 11. The "true" value of the first is to its "assumed value" as "the true value of the second" is to its "assumed value."

$$\therefore \frac{1}{3} \text{ of } 12 : 6 = \frac{1}{4} \text{ of } 20 : (x).$$

Solving, $x = 7\frac{1}{2}$, the true value.

a 12. Mary was half of 24, or 12 years old, when Mary was as old as Ann is now. Therefore Ann is $\frac{1}{2}$ of $(24 + 12)$, or 18.

a 13. Let 100 % = the remainder.

Then by the condition of the problem, B's share being 90%, we have

$$90\% = \$2000 + 10\%$$

$$\text{or } 80\% = \$2000$$

$$1\% = \$25$$

$$100\% = \$2500, \text{ the remainder.}$$

$$\therefore \$2000 + \$2500 = \$4500, \text{ the sum.}$$

a 14. Take A's share as a unit. Then A's share would be 1, D's 6, B's 3, and C's 3. The sum of their shares would be 13. C would therefore receive $\frac{3}{13}$ of \$14.00 or \$3.23.

a 15. Let
 $100\% = \text{cost of the horse.}$
 $66\frac{2}{3}\% = \text{cost of the cow.}$
 $125\% = \text{selling price of the horse.}$
 $16\frac{2}{3}\% \text{ of } 66\frac{2}{3}\% \text{ or } 16\frac{2}{3}\% = \text{loss on cow.}$

Then
 $50\% = \text{selling price of cow.}$
 $175\% = \text{selling price of both.}$
 $175\% = \$210.$

$$1\% = \frac{1}{175} \text{ of } \$210 = \$1.20.$$

$$100\% = \$120, \text{ cost of the horse.}$$

$$66\frac{2}{3}\% = \$80, \text{ cost of the cow.}$$

$$\therefore \text{gain} = (\$120 + \$80) - \$200, \text{ or } \$10.00.$$

a 16. If $\frac{5}{8}$ of the goods were sold for $\frac{3}{4}$ or 75% of the cost, all of the goods at the same rate would bring 120% of the cost. Hence the rate of gain was 20%.

a 17. (a) Each must walk halfway around his track to be again on the straight line. The time required for each to walk this distance is

$$\text{A's time} = (\frac{1}{2} \times \frac{1}{5}) \div 3\frac{1}{2} = \frac{1}{35} \text{ hour.}$$

$$\text{B's time} = (\frac{1}{2} \times \frac{1}{4}) \div 4 = \frac{1}{32} \text{ hour.}$$

$$\text{C's time} = (\frac{1}{2} \times \frac{3}{8}) \div 5 = \frac{3}{80} \text{ hour.}$$

The L. C. M. of these fractions, or 3 hours, is the shortest time elapsing before the three are on the straight line at the same time.

(b) The times required for A, B, and C to walk entirely around are $\frac{2}{35}$, $\frac{1}{16}$, and $\frac{3}{40}$ hour respectively. The L. C. M. of these fractions, or 6 hours, is the shortest time elapsing before the three are at their respective starting points at the same moment.

a 18. The L. C. M. of the numbers 2 to 12 inclusive is $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$, or 27,720. The required number is therefore of the form $27,720n + 1$.

Dividing by 13, we have,

$$\frac{27720n + 1}{13} = 2132n + \frac{4n + 1}{13}.$$

If the number is to be divisible by 13, then $4n + 1$ must be a multiple of 13, and $n = 3$ is obviously the smallest integral value of n which will satisfy this condition.

$\therefore$ required number $= 27720 \times 3 + 1 = 83161$.

a 19. They share in the ratio of 140, 100, and 80, or 7, 5, and 4. Then, C is entitled to $\frac{1}{4}$ of the face of the stone, or $\sqrt{\frac{1}{4}} = \frac{1}{2}$ of the radius, or 8 inches.

a 20. Since similar solids are to each other as the cubes of any two corresponding dimensions, we have

$$\sqrt[3]{\left(\frac{2}{3}\right) \text{ of } 6^3} = 5.24 \text{ in.} = \text{diameter after 1st winds.}$$

$$\sqrt[3]{\left(\frac{1}{3}\right) \text{ of } 6^3} = 4.16 \text{ in.} = \text{diameter after 2nd winds.}$$

$\therefore$ 1st winds off 6 in. $- 5.24$ in., or .76 in.

2nd winds off 5.24 in. $- 4.16$ in., or 1.08 in.

3rd has the ball left or 4.16 in. of the diameter.

$$\begin{aligned} \text{a 21. The \% of married couples} &= \frac{2236}{2501} \times \frac{2374}{2611} \times 100\% \\ &= 81.29\%. \end{aligned}$$

$$\begin{aligned} \text{The \% of widowers} &= \frac{2236}{2501} \times \frac{237}{2611} \times 100\% \\ &= 8.12\%. \end{aligned}$$

$$\begin{aligned} \text{The \% of widows} &= \frac{265}{2501} \times \frac{2374}{2611} \times 100\% \\ &= 9.63\%. \end{aligned}$$

The remaining .96% represents the deceased couples.

ALGEBRA

$$\begin{aligned} \text{b 1. (a)} \quad 1 + 4x^4 &= 1 + 4x^2 + 4x^4 - 4x^2 \\ &= (1 + 2x^2 + 2x)(1 + 2x^2 - 2x). \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad x^{3e} - y^{-3e} &= (x^e)^3 - (y^{-e})^3 \\ &= (x^e - y^{-e})(x^{2e} + x^e y^{-e} + y^{-2e}). \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad e^{3x} - 2e^x - 24e^{-x} &= e^{-x}(e^{2x} + 4)(e^{2x} - 6) \\ &= (e^x + 4e^{-x})(e^{2x} - 6). \end{aligned}$$

$$\text{(d)} \quad e^{3x} + e^{2x} + e^x + 1 = (e^x + 1)(e^{2x} + 1).$$

b 2. Let x = number of gallons of 95% alcohol
and y = number of gallons of 15% alcohol.

$$\text{Then, } \frac{.95x + .15y}{10} = \frac{45}{100} \quad (1)$$

$$\text{and } x + y = 10. \quad (2)$$

$$\text{Solving, } x = 3\frac{3}{4}, \text{ and } y = 6\frac{1}{4}.$$

$$\text{b 3. } 2^{6n+3} \cdot 4^{3n+6} = (8^n)^n$$

$$\text{or } 2^{6n+3} \cdot 2^{6n+12} = 2^{3n^2}$$

$$\text{Then, } 12n + 15 = 3n^2$$

$$\text{or } n^2 - 4n - 5 = 0$$

$$\therefore n = 5, \text{ or } -1.$$

$$\text{b 4. } [(e^x - e^{-x})^2 + 4]^{\frac{1}{2}} = [e^{2x} + 2 + e^{-2x}]^{\frac{1}{2}} = e^x + e^{-x}.$$

$$\begin{aligned} \text{b 5. } \sqrt[3]{192} - 4\sqrt[3]{24} + \sqrt[3]{375} &= 4\sqrt[3]{3} - 8\sqrt[3]{3} + 5\sqrt[3]{3} \\ &= \sqrt[3]{3}. \end{aligned}$$

$$\begin{aligned} \text{b 6. } 3\sqrt{\frac{2}{7}} + 3\sqrt{\frac{7}{2}} - 2\sqrt{\frac{1}{14}} &= \frac{3}{7}\sqrt{14} + \frac{3}{2}\sqrt{14} - \frac{1}{7}\sqrt{14} \\ &= \frac{2.5}{14}\sqrt{14}. \end{aligned}$$

$$\begin{aligned} \text{b 7. } \frac{3}{\sqrt{5}} - \sqrt{2} + \sqrt{7 + 2\sqrt{10}} &= \frac{3}{\sqrt{5}} - \sqrt{2} + \sqrt{2 + 2\sqrt{2}\sqrt{5} + 5} \\ &= \frac{3}{\sqrt{5}} - \sqrt{2} + \sqrt{(\sqrt{2} + \sqrt{5})^2} \\ &= \frac{3}{\sqrt{5}} - \sqrt{2} + \sqrt{2} + \sqrt{5} \\ &= \frac{3}{\sqrt{5}} + \frac{5}{\sqrt{5}} = \frac{8}{\sqrt{5}} = \frac{8\sqrt{5}}{5}. \end{aligned}$$

b 8. By multiplying the fraction by $\frac{2^{\frac{3}{2}}}{2^{\frac{3}{2}}}$ the expressions in parentheses become perfect squares, thus

$$\frac{(8 + 2\sqrt{15})^{\frac{3}{2}} + (8 - 2\sqrt{15})^{\frac{3}{2}}}{(12 + 2\sqrt{35})^{\frac{3}{2}} - (12 - \sqrt{35})^{\frac{3}{2}}} = \frac{(\sqrt{5} + \sqrt{3})^3 + (\sqrt{5} - \sqrt{3})^3}{(\sqrt{7} + \sqrt{5})^3 - (\sqrt{7} - \sqrt{5})^3}.$$

Expanding the expressions and collecting, we have

$$\frac{2[(\sqrt{5})^3 + 3\sqrt{5}(\sqrt{3})^2]}{2[3(\sqrt{7})^2\sqrt{5} + (\sqrt{5})^3]} = \frac{5\sqrt{5} + 9\sqrt{5}}{21\sqrt{5} + 5\sqrt{5}} = \frac{7}{13}.$$

b 9. Let x = number of minutes past noon when A starts
and y = number of minutes past noon when B starts.

Then $\frac{2}{3}(240 - x)$ = time it takes A to arrive at B's position

and $\frac{2}{3}(240 - y)$ = time it takes B to arrive at A's position.

Hence, $x + \frac{2}{3}(240 - x) = y + 11$

and $y + \frac{2}{3}(240 - y) = x + 15.$

Solving for x and y , we get

$x = 219$ minutes past noon, or 3 : 39 P.M.

$y = 222$ minutes past noon, or 3 : 42 P.M.

b 10. Let x = length of train in feet.

The men are walking at the rate of 4.4 feet per second.
The man going in opposite direction to the train goes 22 feet while the train is passing and the other man goes 26.4 feet.

The train goes $x - 22$ feet and $x + 26.4$ feet respectively in passing the men.

$$\therefore \frac{x - 22}{x + 26.4} = \frac{5}{6}$$

$$\text{or } x = 264.$$

b 11. Let a = year in which he was born

Then, $x^2 - x - a = 0,$ (1)

where a is an integer satisfying

$$1800 < a < 1876. \quad (2)$$

Let m and $-n$ be the roots of equation (1).

Then $mn = a$ (3)

and $m - n = 1.$ (4)

Evidently $m > \sqrt{a} > \sqrt{1800} > 42$

and $n < \sqrt{a} < \sqrt{1876} < 44$

Then either $m = 43$ and $n = 42$, by equation (4),

or $m = 44$ and $n = 43.$

The first set of values is consistent with (2) and (3).

$\therefore a = 1806$, and the man was 69 years old in 1875.

b 12. Let d = diameter of the shell

and d = diameter of sphere required to fill the shell.

Then, we have

$$D^3 : d^3 = 1 : 1 - \frac{91}{216}$$

or $D^3 : d^3 = 1 : \frac{127}{216}.$

Extracting the cube root, $D : d = 1 : \frac{5}{6}$, or $d = \frac{5}{6} D.$

$\therefore$ the thickness of the shell $= \frac{1}{2}(D - d),$
 $= \frac{1}{2}(1 - \frac{5}{6}) = \frac{1}{12}$ ft., or 1 in.

b 13. Let x = A's age,

y = B' age.

It is $x - y$ years since A was B's present age. At that time B's age was $y - (x - y)$, or $2y - x$ years.

$$\therefore x = 3(2y - x)$$

or $x = \frac{3}{2}y.$

In 24 years B's age will be $y + 24.$

It is $y - \frac{x}{2}$ years since B was that age.

At that time A's age was $x - (y - \frac{x}{2})$, or $\frac{3x}{2} - y.$

$$\therefore y + 24 = 2\left(\frac{3x}{2} - y\right).$$

Substituting the value of x , we have

$$y + 24 = 2\left(\frac{3}{4}y - y\right).$$

$$\therefore y = 16, \text{ and } x = 24.$$

b 14. Let

x = the part that is copper.

$1 - x$ = the part that is tin.

Weight of the body $= x(8.96) + (1 - x)(7.29).$

$$\therefore [8.96x + 7.29(1 - x)]1.09 = 8.96(1 - x) + 7.29x.$$

Solving, $x = .2905.$

Then copper = 29% and tin = 71%.

b 15. $2^x + 2^{x-1} = 10$, or $2^x(1 + 2^{-1}) = 10$.
 $2^x = 10(\frac{2}{3})$.

Then $x \log 2 = \log 10 + \log 2 - \log 3$
 $= 1 - \log 3 + \log 2$.

$$\therefore x = \frac{1 - \log 3}{\log 2} + 1 = \frac{.522878}{.30103} + 1$$

$$= 2.737.$$

b 16. Let a, b, c , and d denote the weight of the four pieces.
 Then, $a + b + c + d = 40$

$$b - 2a = 1, \text{ or } b = 2a + 1.$$

$$c - 2b - 2a = 1, \text{ or } c = 2b + 2a + 1$$

$$= 6a + 3.$$

$$d - 2c - 2b - 2a = 1, \text{ or } d = 2c + 2b + 2a + 1$$

$$= 18a + 9.$$

$$\therefore a + b + c + d = 27a + 13 = 40$$

$$\therefore a = 1, b = 3, c = 9, \text{ and } d = 27.$$

b 17. Let A = original loan, B = amount of each monthly installment, and R = the monthly rate of interest, $R > 0$.

Then the amount remaining after n installments have been paid is $A(1 + R)^n - BR^{-1}(1 + R)^n + BR^{-1}$. (1)

If the debt is to be fully paid by n installments, then (1) must be zero, and

$$(1 + R)^n = \frac{1}{(1 - AB^{-1}R)}. \quad (2)$$

In the example $A = \$1200$, $B = \$12.50$ and $n = 120$.

$$\therefore (1 + R)^{120} = \frac{1}{(1 - 96R)}$$

or $120 \log (1 + R) + \log (1 - 96R) = 0$.

$$\therefore R = \text{approximately } \frac{1}{2860}.$$

$$\therefore \text{annual rate of interest} = \frac{12}{2860}, \text{ or } 4.6\%.$$

NOTE. If R is given, n can be found by taking the logarithm of both sides of (2).

b 18. The solution given here was sent to the author in 1913 by Dr. Artemas Martin of Washington, D. C.

Let $m^2 - n^2$, $2mn$, $m^2 + n^2$; $m^2 - p^2$, $2mp$, $m^2 + p^2$; $m^2 - q^2$, $2mq$, $m^2 + q^2$ be the respective sides of the three triangles whose areas are to be equal. Then must

$$mn(m^2 - n^2) = mp(m^2 - p^2) = mq(m^2 - q^2)$$

or $n(m^2 - n^2) = p(m^2 - p^2) = q(m^2 - q^2)$;
 and we must have

$$n(m^2 - n^2) = p(m^2 - p^2) \quad (1)$$

$$n(m^2 - n^2) = q(m^2 - q^2) \quad (2)$$

$$p(m^2 - p^2) = q(m^2 - q^2). \quad (3)$$

From (1), by transposition,

$$(n - p)m^2 = n^3 - p^3. \quad (4)$$

Dividing by $(n - p)$,

$$m^2 = n^2 + np + p^2. \quad (5)$$

In (5), put $n = r^2 - s^2$, $p = 2rs + s^2$, and we get

$$m = r^2 + rs + s^2.$$

Comparing (1) and (3), (2) and (3),

we have $q^3 - m^2q = -n(m^2 - n^2). \quad (6)$

$$q^3 - m^2q = -p(m^2 - p^2). \quad (7)$$

Also, $q^3 - m^2q = -q(m^2 - q^2). \quad (8)$

In these three equal cubic equations n, p, q are the roots, and since the second term is wanting, q may be taken = $-(n + p)$, = $r^2 + 2rs$; for the positive value will equally satisfy the third equation, and the values of n and p will satisfy the first and second equations.

Substituting the general values of m, n, p, q in the supposition we started out with, the sides of the three required triangles are

$$(r^2 + rs + s^2)^2 - (r^2 - s^2)^2, \quad 2(r^2 - s^2)(r^2 + rs + s^2).$$

$$(r^2 + rs + s^2)^2 + (r^2 - s^2)^2;$$

$$(r^2 + rs + s^2)^2 - (2rs + s^2)^2, \quad 2(2rs + s^2)(r^2 + rs + s^2),$$

$$(r^2 + rs + s^2)^2 + (2rs + s^2)^2;$$

$$(r^2 + rs + s^2)^2 - (r^2 + 2rs)^2, \quad 2(r^2 + 2rs)(r^2 + rs + s^2),$$

$$(r^2 + rs + s^2)^2 + (r^2 + 2rs)^2;$$

where r may be any number, and s any number less than r .

Examples. 1. Take $r = 2$, $s = 1$, and we have the triangles 40, 42, 58; 24, 70, 74; 15, 112, 113; each area = 840.

2. Take $r = 3$, $s = 1$, and we get the three triangles 56, 390, 394; 105, 208, 233; 120, 182, 218, neglecting the negative sign. The areas each = 10,920.

3. Take $r = 3$, $s = 2$, and we find the three triangles 190, 336, 386; 105, 608, 617; 80, 798, 802. Areas = 31,920. The number of different sets of three such triangles is infinite.

b 19. Factoring, we have

$$(2^x - 16)(2^x - 5) = 0.$$

$$\therefore 2^x = 16 \text{ and } 2^x = 5.$$

$$\therefore 2^x = 2^4, \text{ or } x = 4.$$

Also, $x \log 2 = \log 5.$

$$\therefore x = \frac{\log 5}{\log 2} = \frac{.6990}{.3010} = 2.322.$$

$$\therefore x = 4, \text{ or } 2.322.$$

b 20. Let $6x$ = mother's age when she was 3 times as old as the monkey was.

Then $2x$ = monkey's age at that time.

$4x$ = the difference in their ages.

$18x$ = age the monkey will be when 3 times, etc.

$9x$ = age of the mother when $\frac{1}{2}$ as old as monkey was, when, etc.

$9x - 4x$, or $5x$ = age of monkey at same time.

$10x$ = mother's age now.

$6x$ = monkey's age now.

$10x + 6x$, or $16x = 4$, their combined ages

$$\therefore x = \frac{1}{4} \text{ yr.}$$

$\therefore$ mother is $2\frac{1}{2}$ yrs. old.

$\therefore$ weight and monkey weigh $2\frac{1}{2}$ lb. each.

Let R = weight of rope.

Then, $2\frac{1}{2} + R = \frac{3}{2}(2\frac{1}{2} + 2\frac{1}{2} - 2\frac{1}{2}).$

$$\therefore R = \frac{3}{4} \text{ lb.}$$

Length of rope = $(\frac{3}{4} \times 16) \div 4 = 5.$

$\therefore$ length of rope = 5 ft.

b 21. (a) For the distance series, we have

$$S = 50 + 2 \times 25 \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \frac{1}{2^n} \right).$$

$\therefore S = 50 + 100$, or 150 ft., for an infinite number of terms.

(b) For the time series, we have

$$T = \left(\frac{50}{16.1} \right)^{\frac{1}{2}} + 2 \left(\frac{25}{16.1} \right)^{\frac{1}{2}}$$

$$\left[1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \dots \left(\frac{1}{2^n} \right)^{\frac{1}{2}} \right].$$

$\therefore T = \left(\frac{25}{16.1} \right)^{\frac{1}{2}} [\sqrt{2} + 4 + 2\sqrt{2}]$, or 10.27 sec. for an infinite number of terms.

b 22. The equation may be written in the form

$$x^{1+\frac{1}{x}} = x^x$$

or

$$\frac{x+1}{x^x} - x^x = 0.$$

$$\therefore x^x \left[x^{\frac{(x+1-x^2)}{x}} - 1 \right] = 0.$$

$$\therefore x^x = 0 \text{ and } x^{\frac{x+1-x^2}{x}} - 1 = 0.$$

The first of these equations is satisfied for the value of $x = -\infty$.

From the second equation, we have, by taking logarithms,

$$\left(\frac{x+1-x^2}{x} \right) \log x = 0,$$

which is equivalent to the three equations

$$\frac{1}{x} = 0, \quad x+1-x^2 = 0, \text{ and } \log x = 0.$$

Solving the first, $x = \pm \infty$.

Solving the second, $x = \frac{1}{2}(1 \pm \sqrt{5}).$

Solving the third, $x = 1.$

By substituting these values in the original equation, we find that 0 and $\pm \infty$ are to be rejected.

Hence the roots are 1 and $\frac{1}{2}(1 \pm \sqrt{5}).$

PLANE GEOMETRY

c 1. Draw the lines of centers OO' , OO'' , and $O'O''$. Also draw $O'E$ and $O''D$.

Then
 OO' passes through A ,
 $O'O''$ passes through B ,
 and
 OO'' passes through C .

Then
 $\angle n = \angle n_1$. (being opposite equal sides)

$\angle n_1 = \angle n_2$. (being vertical $\angle$ s)

$\angle n_2 = \angle n_3$. (being opposite equal sides)

$\therefore \angle n = \angle n_3$.

$\therefore O'E$ is $\parallel$ to OO' .

Likewise

$\angle m = \angle m_3$

and $O''D$ is $\parallel$ to OO' .

$\therefore EO''D$ is a straight line

and the chord ED is a diameter.

NOTE. $EO''D$ may also be proven a straight line by showing that the sum of the $\angle$ s at O'' = the sum of the $\angle$ s of the $\triangle O'OO''$.

c 2. Let H be the intersection of the altitudes.

$CDHF$ is concyclic.

(two opposite $\angle$ s are rt. $\angle$ s)

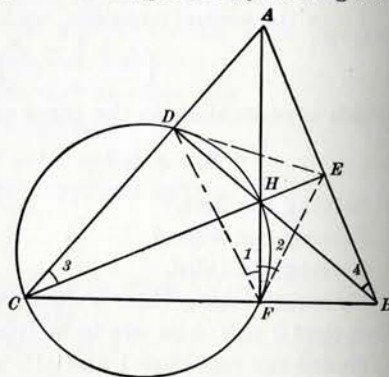
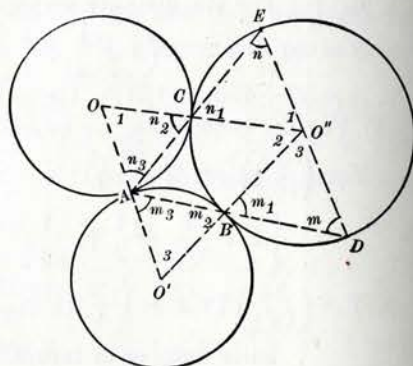
$\therefore \angle 1 = \angle 3$.

Likewise $\angle 2 = \angle 4$.

But $\angle 3 = \angle 4$.

(being complements of $\angle A$)

$\therefore \angle 1 = \angle 2$.



c 3. Analysis. Produce CD to C' , making $CD = C'D$.

Then $\triangle CC'B$ is determined,

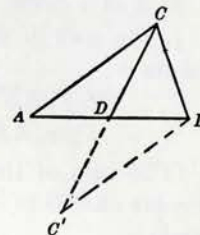
for $CB = a$, $BC' = AC = b$,

$CC' = 2m_c$.

Construction. Construct $\triangle CC'B$; draw the median DB and produce it to A , making $AD = DB$.

Draw AC .

Then ABC is the required triangle.



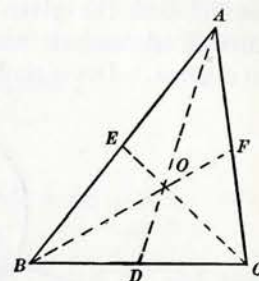
c 4. Solution I. Analysis. The $\triangle BOC$ is determined since in this $\triangle$ two sides and the median to the third side are known.

$BO = \frac{2}{3}m_b$, $OC = \frac{2}{3}m_c$, $OD = \frac{1}{3}m_a$.

Construction. Construct $\triangle OBC$ by means of the previous exercise. Produce BO to F , making $OF = \frac{1}{2}BO$, and CO to E , making $OE = \frac{1}{2}CO$.

Draw BE and CF , and produce them to meet at A .

Then ABC is the required triangle.



Solution II. With $\frac{2}{3}$ of each median construct a triangle OBG . Bisect OG . Draw BF . Extend BF its own length to C . Extend $BO \frac{1}{2}$ its own length to E . Extend CE its own length to A . Join A and B . ABC is the required triangle.

c 5. Semicircle on DC = semicircle on AD - semicircle on AC . (1)

Semicircle on DC = semicircle on DB - semicircle on BC . (2)

But since semicircle on AD + semicircle on DB = semicircle on AB , we obtain by addition of (1) and (2), the following,

Circle on DC = semicircle on AB - (semicircle on AC + semicircle on BC).

c 6. Draw the diameter AOF of the given circle.

Area of a circle = $\pi r^2 = \frac{1}{4} \pi d^2$.

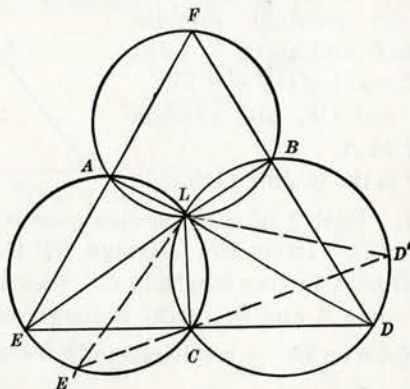
$\therefore$ the sum of the areas of $\odot$ described on the four segments

$$= \frac{1}{4} \pi \overline{AE}^2 + \frac{1}{4} \pi \overline{EB}^2 + \frac{1}{4} \pi \overline{CE}^2 + \frac{1}{4} \pi \overline{ED}^2 \\ = \frac{1}{4} \pi (\overline{AE}^2 + \overline{EB}^2 + \overline{CE}^2 + \overline{ED}^2) = \frac{1}{4} \pi \overline{AF}^2.$$

(The sum of the squares of the segments of two perpendicular chords is equal to the square of the diameter of the circle.)

$\therefore$ area given $\odot$ = area $\odot AE$ + area $\odot EB$ + area $\odot CE$ + area $\odot ED$.

c 7.* Let the given points be A, B, C . Draw a circle through A and B such that AB subtends an angle of 60 degrees. Draw similar circles for BC and AC . Then the



largest equilateral triangle is DEF whose sides are perpendicular to the lines joining A, B, C to the point L which is common to the three circles. Let $E'D'$ be the side of any other triangle. Then the triangle $E'D'L$ is similar to triangle EDL . But since EL and DL are diameters, $E'L$ and $D'L$ are shorter, and hence $E'D'$ is shorter than EF .

* From "School Science and Mathematics."

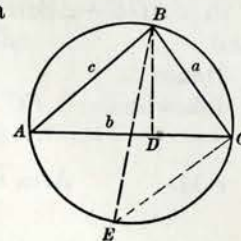
c 8. Let ABC be the triangle, BE a diameter, and BD the altitude.

Then $\triangle ADB$ and BCE are similar.

$$\therefore AB : EB = BD : BC$$

$$\text{or, } EB \cdot BD = ac \text{ and } BD = \frac{ac}{2R}.$$

$$\therefore \text{area } \triangle ABC = \frac{1}{2} b(BD) \\ = \frac{1}{2} b \left(\frac{ac}{2R} \right) = \frac{abc}{4R}.$$



c 9. Draw the lines AC and BD and let d = diameter.

Then $AC \cdot BD = CD \cdot AB + CB \cdot AD$.

(In an inscribed quadrilateral, the product of the diagonals is equal to the sum of the products of the opposite sides.)

$\triangle DBA$ and ACD are rt $\triangle$.

$$\therefore (x^2 - 10^2)^{\frac{1}{2}} = AC$$

and

$$(x^2 - 6^2)^{\frac{1}{2}} = BD.$$

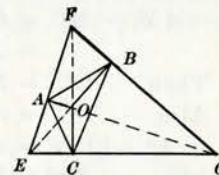
$$\therefore (x^2 - 10^2)^{\frac{1}{2}} (x^2 - 6^2)^{\frac{1}{2}} = AC \cdot BD = 60 + 8x$$

$$\therefore x = 16.1116 \text{ and } r = 8.0558.$$

c 10. Join the three given points A, B , and C , and construct AO, BO , and CO , the three bisectors of $\angle A, B$, and C .

At A, B , and C draw perpendiculars upon AO, BO , and CO , respectively, and produce them until they meet E, F , and G . EFG is the $\triangle$ required.

Proof. Draw OE, OF , and OG . $AOCE$ is concyclic.



(Each angle formed by joining the feet of the three altitudes of a triangle is bisected by the corresponding altitude.)

$$\therefore \angle ACO = \angle AEO. \text{ Similarly } \angle BCO = \angle OGB.$$

$$\therefore \angle AEO = \angle OGB.$$

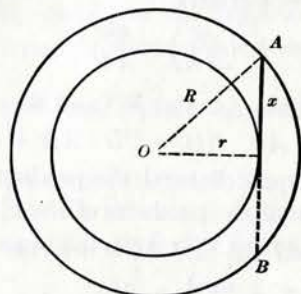
$$\therefore \angle AEO = \angle BOG. \text{ Similarly } \angle AOF = \angle COG.$$

and

$$\angle FOB = \angle EOC.$$

$\therefore \angle AOE + \angle AOF + \angle FOB = 180^\circ$,
and EB is a straight line.
Hence EB is an altitude.
Likewise FC and AG are altitudes.
 $\therefore EFG$ is the required $\triangle$.

c 11. Area of the outer circle $= \pi R^2$.



Area of the inner circle $= \pi r^2$.

$\therefore$ area of ring $= \pi(R^2 - r^2) = \pi x^2$.

c 12. Let R and r denote the radii of the outer and inner circles respectively. Also let $x = AM$ and $t =$ the tangent QP .

When the chord is constructed we will have,

$$AM = MC = CN = NB$$

$$= x.$$

$$\text{Then } \overline{OC}^2 = R^2 - (2x)^2.$$

$$\text{Also } \overline{OC}^2 = r^2 - x^2.$$

$$\therefore R^2 - (2x)^2 = r^2 - x^2$$

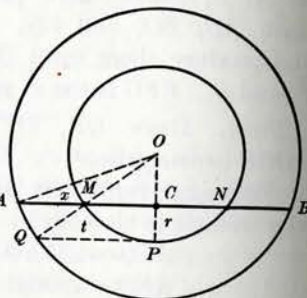
$$\text{or } 3x^2 = R^2 - r^2.$$

$$\text{Also } \overline{OP}^2 = t^2 = R^2 - r^2.$$

$$\therefore 3x^2 = t^2$$

$$\text{or, } x = \sqrt{t \cdot \frac{t}{3}}.$$

$\therefore x$ must be the mean proportional between t and $\frac{t}{3}$.



c 13. By Ex. 1 MN is a diameter of circle CB . $\therefore \angle ACN$ is a rt. $\angle$ being inscribed in a semi-circle.

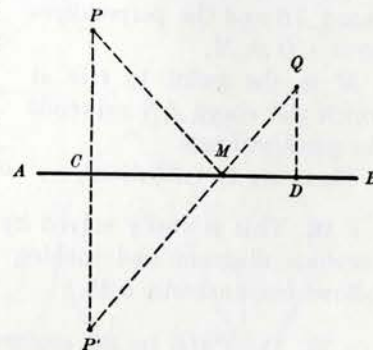
c 14. By Ex. 13 $\angle MCN$ and $\angle MBN$ are rt. $\angle$ s. $\therefore \angle ACR$ and $\angle ABR$ are rt. $\angle$ s, being supplements of rt. $\angle$ s.

$\therefore$ the vertices of the quadrilateral $ABRC$ are concyclic.

c 15. Draw $PCP' \perp AB$ making $PC = P'C$. Join P' and Q cutting AB at M . Then M is the required point.

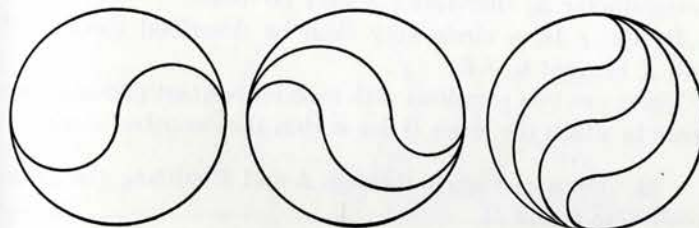
The minimum sum of distances from P to Q is given by

$$(PM + QM)^2 = \overline{PQ}^2 + 4PC \cdot QD.$$



c 16. Find P_1 , the reflection of P at the first surface; then P_2 , the reflection of P_1 at the second surface; next P_3 , the reflection of P_2 at the third surface and so on if there be more surfaces. Finally draw P_3Q , cutting CD in M ; draw MP_2 , cutting BC in N , and draw NP_1 , cutting AB in O . Then $PONMQ$ is the required path.

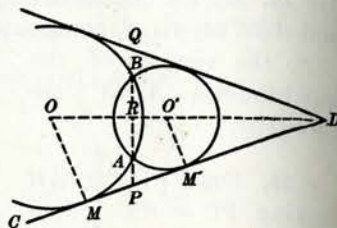
c 17.



c 18. Let the circle whose center is O be the required circle tangent to CD at M . Then $\overline{PM}^2 = \overline{PB} \cdot \overline{PA}$. Hence the point M can be found, and the center of the required circle is the intersection of the perpendicular bisector of the chord AB and the perpendicular to CD at M .

M is the point in CD at which the chord AB subtends the greatest angle.

There are two solutions.



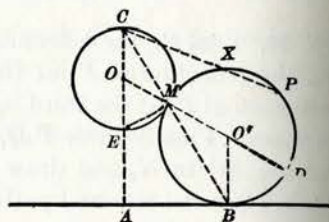
c 19. This is easily solved by drawing $PRQ \perp OD$ in the previous diagram and making $RA = RB$. The solution follows from exercise c 18.

c 20. Let PMB be the required circle. Draw $OA \perp AB$. Draw OO' and produce to D . Let M be the point of contact. Draw MC , ME , MB , and CP , cutting circle O' in X . Then

$$\angle AOM = \angle DO'B.$$

Hence $\angle OMC = \angle O'MB$

(being halves of equal $\angle$ s) and CM and MB are in the same straight line.



Since, $CX \cdot CP = CM \cdot CB$ and $CM \cdot CB = CE \cdot CA$, being similar Δ s, therefore CX may be found.

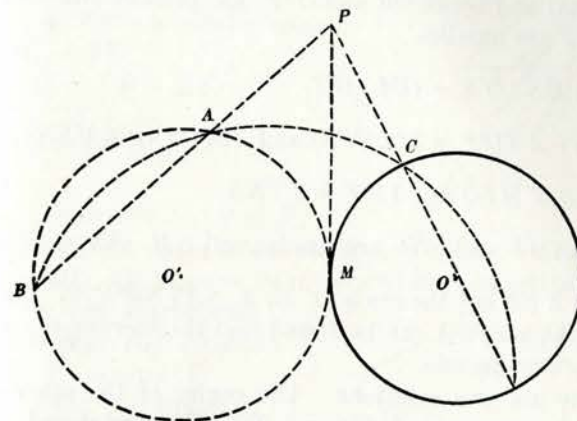
By Ex. c 18, a circle may then be described through P and X tangent to AB .

There are two solutions with exterior contact and also two more in which the circle O lies within the described circle.

c 21. Draw any circle through A and B cutting the given circle O in C and D .

Let the lines AB and CD meet in P . Draw PM tangent to the given circle. Now,

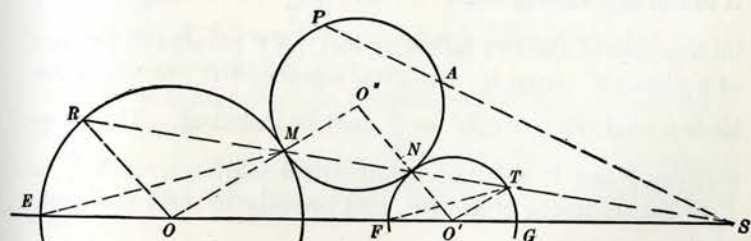
$$PA \cdot PB = PC \cdot PD.$$



A circle drawn through A , B , M will be tangent to both PM and the given circles at M . This circle may be drawn by Ex. c 18.

There are two solutions corresponding to the two tangents from P to the given circle. The point M is the point in PM at which the chord CD subtends the greatest angle.

c 22. Let O'' be the required circle passing through P and tangent to circles O and O' at M and N . Draw OO''



and $O'O''$. Draw MN meeting OO' produced at S and cutting circles O and O' in R and T .

$\angle R = \angle M = \angle N = \angle T$ (being base $\angle$ of similar isosceles triangles) and hence OR and $O'O''$ are parallel and also OM and $O'T$ are parallel.

$$\therefore OS : O'S = OM : O'T.$$

$$\text{Also } \angle TO'S = 2(\angle TFS) \text{ and } \angle MOS = 2(\angle MES).$$

$$\therefore \angle MEO = \angle TFO' = \angle TNG.$$

$$\therefore \triangle SME \text{ and } \triangle SNG \text{ are similar and } SM \cdot SN = SG \cdot SE.$$

Now if SP cut the circle O'' in A , $SA \cdot SP = SG \cdot SE$ and hence the point A can be found and the exercise is reduced to the previous one.

There are two solutions. The center of the other circle is the intersection of RO and TO' when produced. This circle touches the given circles in R and T .

c 23. Let O, I, H be the circumcenter, the incenter, and the orthocenter, respectively. Describe the circle ABC . The perpendicular, OE , to BC is equal to $\frac{AH}{2}$, hence BC is known; also angle A . The radius, r , of the inscribed circle is found graphically from $r = AI \sin \frac{A}{2}$. To locate I find the intersection of the two following loci: XY parallel to BC and at a distance r from it; also, the segment BIC , in which the known angle BIC , or $90^\circ + \frac{A}{2}$, may be inscribed. The intersections I and I' give two symmetrical solutions. Join I or I' to the midpoint of arc BC , and produce to meet the circle at A .

c 24. Let CD be a median of $\triangle ACB$. Produce CD by its own length to E .

Draw AE and BE .

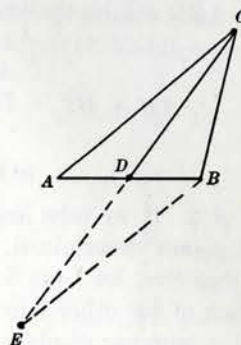
Then $ACEB$ is a parallelogram.

$$\therefore AC = BE.$$

$$\text{But } CE < CB + BE,$$

$$\text{or } CE < CB + AC.$$

$$\therefore CD < \frac{1}{2}(CB + AC).$$



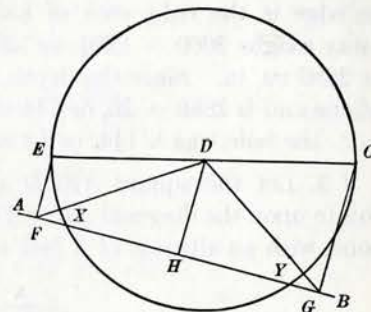
c 25. Draw $DH \perp AB$ at H .

Then EF, DH , and CG are parallel.

$FH = HG$. (If three or more parallel lines intercept equal parts on one transversal, they intercept equal parts on every transversal.)

DH is the perpendicular bisector of FG and $DF = DG$. (Any point on the perpendicular bisector of a line is equidistant from the extremities of the line.)

$\therefore F$ and G are equidistant from the center G .



c 26. Let AM and AN be the two given lines, and P the given point.

Measure off AD on AM and AE on AN so that $AD = AE = S$, one-half length of given line.

Let O be center of circle tangent to AM and AN at D and E , respectively.

Through P draw a line tangent to circle at T and cutting AM in B and AN in C , this tangent line to lie between A and O .

ABC will be the triangle required.

$$AB + BT = AB + BD = s.$$

$$AC + CT = AC + CE = s.$$

$$\therefore AB + BT + TC + AC = AB + BC + AC = 2s$$

SOLID GEOMETRY

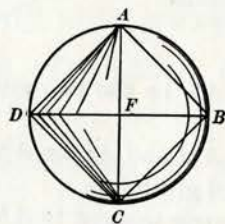
d 1. If we take line 1 with each of the other six, we have 6 planes determined. If we take line 2 with each of the other five, we have 5 more determined. If we take 3 with each of the other four we have 4 more.

$\therefore$ number of planes determined = $6 + 5 + 4 + 3 + 2 + 1$, or 21.

d 2. The block contains $2000 \div \frac{1}{8}$, or 8000 cu. in., and its edge is the cube root of 8000, or 20 in. The part cut away weighs $2000 - 1280$, or 720 lb., and occupied $720 \div \frac{1}{8}$, or 2880 cu. in. Since the depth of the cut is 20 in., the area of one end is $2880 \div 20$, or 144 sq. in.

$\therefore$ the hole was $\sqrt{144}$, or 12 inches square.

d 3. Let the square $ABCD$ and its circumscribing circle rotate upon the diagonal AC . The square generates a double cone, with an altitude of 3 feet and diameter of base 6 feet.



Volume of double cone = $2(9\pi)$, or 18π .

Surface of double cone = $2(9\pi\sqrt{2})$, or $18\pi\sqrt{2}$.

Volume of sphere generated by circle = $\frac{1}{6}\pi(6^3)$, or 36π .

Surface of sphere generated by circle = $4[\frac{1}{4}\pi(6)^2]$, or 36π .

$\therefore$ volume of sphere = 2 times volume of double cone
and surface of sphere = $\sqrt{2}$ times surface of double cone.

d 4. In the similar $\triangle OEC$ and OCB

$$\frac{OE}{OC} = \frac{OC}{OB}$$

That is

$$\frac{OE}{R} = \frac{R}{R+h}$$

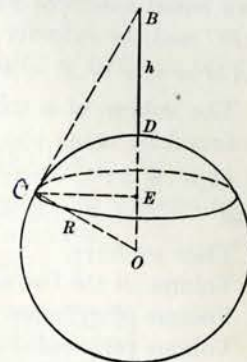
Then

$$OE = \frac{R^2}{R+h}$$

$$DE = R - \frac{R^2}{R+h} = \frac{Rh}{R+h}$$

$\therefore$ area of zone = $Z =$

$$2\pi R \cdot \frac{Rh}{R+h} = \frac{2\pi R^2 h}{R+h}$$



Ex. *d 4*

d 5.

$$Z = \frac{2\pi R^2 h}{R+h}$$

But

$$h = R.$$

$$\therefore Z = \frac{2\pi R^2}{2R} = \pi R^2.$$

Surface of earth = $S = 4\pi R^2$.

$$\therefore \frac{Z}{S} = \frac{\pi R^2}{4\pi R^2} = \frac{1}{4}.$$

d 6.

$$S = 4\pi r^2.$$

Then

$$Z = \frac{S}{6} = \frac{2}{3}\pi R^2.$$

But

$$Z = \frac{2\pi R^2 h}{R+h}$$

Ex. *d 4*

$$\therefore \frac{2\pi R^2 h}{R+h} = \frac{2\pi R^2}{3}$$

$$\therefore h = \frac{1}{2}R.$$

d 7. The diagram represents a vertical section of the sphere.

The volume removed consists of two equal spherical segments such as ABC and the cylinder AP .

$$AM = \sqrt{5^2 - 3^2} = 4 \text{ and } CM = 1.$$

The volume of a spherical segment is found by using the formula

$$\frac{1}{8} \pi h(h^2 + 3r^2), \text{ where } h = CM = 1,$$

and $r = AM = 3$

Then we have,

$$\text{Volume of the two segments} = 2\left[\frac{1}{8} \pi h(h^2 + 3r^2)\right] = 9\frac{1}{3} \pi.$$

$$\text{Volume of cylinder} = 8 \pi AM^2 = 72 \pi.$$

$$\text{Volume removed} = 9\frac{1}{3} \pi + 72 \pi = 81\frac{1}{3} \pi.$$

$$\text{Volume of sphere} = \frac{4}{3} \pi AO^3 = 166\frac{2}{3} \pi.$$

$$\text{Volume remaining} = 166\frac{2}{3} \pi - 81\frac{1}{3} \pi = 85\frac{1}{3} \pi.$$

$$\therefore \text{part of sphere remaining} = 85\frac{1}{3} \pi \div 166\frac{2}{3} \pi = .512.$$

d 8. We shall first prove that the square of the area of a face is the mean proportional between the area of its projection on the base and the base.

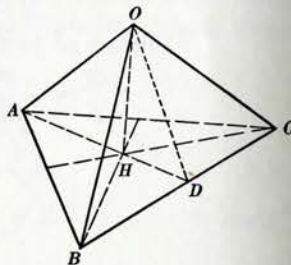
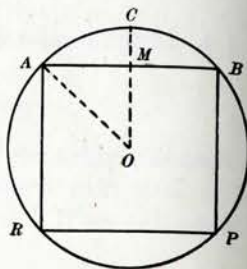
Pass a plane through $OA \perp BC$. It is $\perp$ to the base ABC and therefore contains the altitude OH of the tetrahedron. Let AD be the intersection of this plane with the base.

We then have $OA \perp OD$, $OD \perp BC$, $AD \perp BC$, and $OH \perp AD$.

In the rt. $\triangle AOD$, we have,

$$OD^2 = (AD) \times (HD). \quad (1)$$

Since the $\triangle BOC$, BAC , and BHC have the same base BC , their areas are proportional to their altitudes OD , AD , and HD .



Hence by (1), we have,

$$(BOC)^2 = (BAC) \times (BHC). \quad (2)$$

$$\text{Similarly, } (BOC)^2 = (BCA) \times (BHA). \quad (3)$$

$$(AOC)^2 = (ABC) \times (AHC). \quad (4)$$

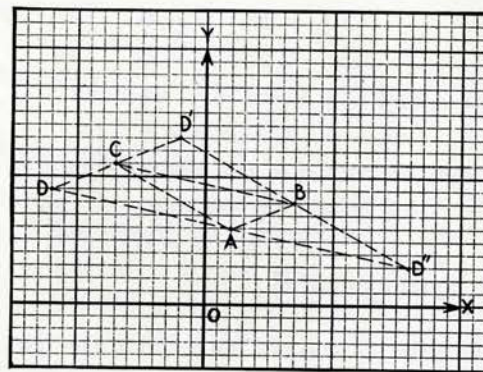
Adding, we get

$$(BOC)^2 + (BOA)^2 + (AOC)^2 = (ABC)(BHC + BHA + AHC) = (ABC)^2.$$

ANALYTICAL GEOMETRY

e 1. Let A , B , and C denote the three given vertices and D , D' , and D'' the three other possible vertices.

$$\text{Slope of } DC = \text{slope of } AB = \frac{2}{3}.$$



Then the equation of DC is

$$y - 11 = \frac{2}{3}(x + 7).$$

$$\therefore 5y - 2x = 69.$$

$$\text{Slope of } AD = \text{slope of } BC = -\frac{3}{4}.$$

Then the equation of AD is

$$y - 6 = -\frac{3}{4}(x - 2).$$

$$\therefore 14y + 3x = 90.$$

Solving equations $5y - 2x = 69$ and $14y + 3x = 90$, we find the coördinates of D to be $(-12, 9)$.

Slope of BD' = slope of $AC = -\frac{5}{9}$.

Then the equation of BD' is

$$y - 8 = -\frac{5}{9}(x - 7).$$

$$\therefore 9y + 5x = 107.$$

Solving equations $9y + 5x = 107$ and $5y - 2x = 69$, we find the coördinates of D' to be $(-2, 13)$.

Solving equations $14y + 3x = 90$ and $9y + 5x = 107$, we find the coördinates of D'' to be $(16, 3)$.

e 2. The equation of any line is $Ax + By + C = 0$. If this line passes through the given points, the coördinates of each of the points must satisfy the equation.

$$\text{Hence,} \quad 2A + B + C = 0$$

$$\text{and} \quad 3A - 2B + C = 0.$$

Solving for A and B in terms of C , we have

$$A = -\frac{3}{7}C \text{ and } B = -\frac{1}{7}C.$$

Substituting these values in the original equation, we have

$$-\frac{3}{7}Cx + \frac{1}{7}Cy + C = 0$$

$$\text{or} \quad 3x + y - 7 = 0,$$

which is the equation desired.

e 3. Putting the equation of the chord in the normal form, we have

$$\frac{b}{\sqrt{a^2 + b^2}}x + \frac{a}{\sqrt{a^2 + b^2}}y = \frac{ab}{\sqrt{a^2 + b^2}}.$$

$$\therefore \text{distance of the chord from the center} = \frac{ab}{\sqrt{a^2 - b^2}}.$$

Since this distance, one half of the chord, and the radius of the circle form a right triangle, of which r is the hypotenuse,

$$\therefore \text{length of chord} = 2\left[r^2 - \frac{a^2b^2}{a^2 + b^2}\right]^{\frac{1}{2}}.$$

e 4. Reducing the equation to the normal form, we have

$$\frac{a}{\sqrt{a^2 + b^2}}x + \frac{b}{\sqrt{a^2 + b^2}}y = \frac{c^2}{\sqrt{a^2 + b^2}}.$$

$$\therefore \text{distance from the origin} = \frac{c^2}{\sqrt{a^2 + b^2}}.$$

e 5. Let $x^2 + y^2 + 2Dx + 2Ey + F = 0$ be the equation of the circle. It passes through the three points (o, o) , (a, o) , and (o, b) .

Substituting the coördinates of each of the points in the given equation, we have

$$F = 0,$$

$$a^2 + 2aD + F = 0,$$

$$b^2 + 2bE + F = 0.$$

Solving, we get

$$F = 0, D = -\frac{1}{2}a, E = -\frac{1}{2}b.$$

Substituting these values in the equation of the circle, we find

$$x^2 + y^2 - ax - by = 0,$$

as the required equation.

e 6. Solving the equations for x and y , we find the coördinates of the ends of the common chord to be

$$\left(\frac{1}{2}a, \frac{1}{2}\sqrt{4r^2 - a^2}\right) \text{ and } \left(\frac{1}{2}a, -\frac{1}{2}\sqrt{4r^2 - a^2}\right).$$

The center of the required circle is the mid-point of the chord or $(\frac{1}{2}a, o)$ and the radius is $\frac{1}{2}\sqrt{4r^2 - a^2}$. Substituting the values of a , b , and r in $(x - a)^2 + (y - b)^2 = r^2$, we have

$$(x - \frac{1}{2}a)^2 + y^2 = \frac{1}{4}(4r^2 - a^2).$$

$$\therefore \text{the required equation} = x^2 - ax + y^2 = r^2 - \frac{1}{2}a^2.$$

e 7. Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ be the vertices of the triangle. The slope of the line P_1P_2 is $\frac{(y_1 - y_2)}{(x_1 - x_2)}$ and the coördinates of the mid-point of P_1P_2 are $\frac{(x_1 + x_2)}{2}$, $\frac{(y_1 + y_2)}{2}$.

The equation of the perpendicular bisector of P_1P_2 is, therefore,

$$\left(y - \frac{y_1 + y_2}{2}\right) = -\frac{x_1 - x_2}{y_1 - y_2} \left(x - \frac{x_1 + x_2}{2}\right)$$

or

$$2(x_1 - x_2)x + 2(y_1 - y_2)y - x_1^2 + x_2^2 - y_1^2 + y_2^2 = 0.$$

In like manner, it will be found that the equations of the perpendicular bisectors of P_2P_3 and P_3P_1 are, respectively,

$$2(x_2 - x_3)x + 2(y_2 - y_3)y - x_2^2 + x_3^2 - y_2^2 + y_3^2 = 0,$$

$$2(x_3 - x_1)x + 2(y_3 - y_1)y - x_3^2 + x_1^2 - y_3^2 + y_1^2 = 0.$$

If these three equations be added, the left member of the new equation vanishes identically. Hence the three lines meet in a point.

e 8. Let a be the radius of the inscribed circle having its center at the origin. Let K , L , and M be the three points of tangency, having the coördinates (d, e) , (b, c) , and $(o, -a)$. Then the vertices of the triangle, A , B , and C , have the following coördinates:

$$\left\{\frac{a^2+ac}{b}, -a\right\}, \left\{\frac{a^2(e-c)}{be-cd}, \frac{a^2(b-d)}{be-cd}\right\}, \left\{\frac{a^2+ae}{d}, -a\right\}.$$

The equations of the lines AK , CL , and BM are as follows:

$$AK: (ab + be)x - (bd - a^2 - ac)y - a(bd + ae + ce) = 0.$$

$$CL: (dc + ad)x - (bd - a^2 - ae)y - a(bd + ce + ac) = 0.$$

$$BM: (ab - ad + be - cd)x - a(e - c)y - a^2(e - c) = 0.$$

The vanishing of the determinant formed from the coefficients of the three preceding equations proves that the three lines AK , CL , and BM are concurrent.

e 9. Let O be the center of the inscribed circle and G the centroid.

Draw the median CD through G ; CH , OF , and GE perpendicular to AB , and GL perpendicular to OF .

Then we have

$$\begin{aligned} AE &= AH + \frac{2}{3}HD = AH + \frac{2}{3}(AD - AH) = \frac{1}{3}AH + \frac{2}{3}AD \\ &= \frac{b^2 + c^2 - a^2}{6c} + \frac{c}{3} = \frac{b^2 + 3c^2 - a^2}{6c}. \end{aligned}$$

Also

$$AF = \frac{b + c - a}{2}.$$

$$\therefore EF = GL = AE - AF = \frac{b^2 - a^2 - 3bc + 3ac}{6c}.$$

Also if S = area of $\triangle ABC$, we have

$$GE = \frac{1}{3}CH = \frac{2S}{3c}.$$

The equation of the inscribed circle, referred to the axes AB and OF , is $x^2 + (y - r)^2 = r^2$

$$\text{or } x^2 + y^2 - 2ry = 0.$$

Now if we substitute the coördinates of G in the expression $x^2 + y^2 - 2ry$ and let $s = \frac{1}{2}(a + b + c)$ we have

$$\left(\frac{b^2 - a^2 - 3bc + 3ac}{6c}\right)^2 + \frac{4S^2}{9c^2} - \frac{4S^2}{3cs},$$

which reduces to

$$c^2[5(a^2 + b^2 + c^2) - 6(ab + ac + bc)]$$

and equals zero if

$$5(a^2 + b^2 + c^2) = 6(ab + ac + bc).$$

$\therefore$ the incircle passes through the centroid of the triangle.

e 10. Given $A(o, o)$, $B(a, o)$, and the equation of CD as $y = c$, the constant length of xy as d , and the coördinates of $P(h, k)$. The line PB cuts the line CD at the point $\left[\left(\frac{ch + ah - ac}{k}, c\right)\right]$, and PA cuts the line CD at the point $\left(\frac{ch}{k}, c\right)$. Hence $\frac{ak - ac}{k} = d$, or $k = \frac{ac}{a - d}$, which is the equation of a line parallel to AB . Two results are obtained according as $a < d$ or $a > d$.

PHYSICS

f 1. The bullet moves in a curve of the parabola BM . By the laws of falling bodies and of projectiles, the bullet would arrive at M in the same time that it would arrive at L if dropped from B .

Therefore the height of the balloon is obtained by the formula for the distance a body falls in a given time, which is $s = \frac{1}{2}gt^2$ where $g = 32.16$, the increment due to gravity, and t the time in seconds.

Then we have

$$\text{height of balloon} = \frac{1}{2}gt^2.$$

Also

$$\text{height of balloon} = vt,$$

where v = velocity of sound per second

$$\therefore \frac{1}{2}gt^2 = vt$$

or,
$$t = \frac{2v}{g}.$$

$$\therefore \text{height} = vt = v\left(\frac{2v}{g}\right) = \frac{2v^2}{32.16} = \frac{v^2}{16.08}.$$

Taking $v = 1090.5$ feet, we have

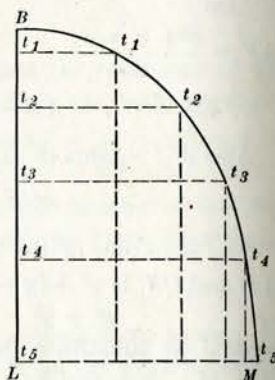
$$\text{height of balloon} = 73,954.61 \text{ ft.}$$

f 2. Using the formula $s = \frac{1}{2}gt^2$, we get $t^2 = \frac{2S}{g}$, or

$$t = \sqrt{\frac{2s}{g}}.$$

Then
$$t = \sqrt{\frac{2s}{g}} = \sqrt{\frac{2 \times 100.5}{32.16}} = \sqrt{\frac{25}{4}} = 2.5.$$

$$\therefore \text{distance from base of cliff} = 2.5 \times 900 \text{ ft., or } 2250 \text{ ft.}$$



$$f 3. C = \frac{5}{9}(F - 32^\circ) \text{ and } F = \frac{9}{5}C + 32^\circ.$$

Placing $F = C$, we have

$$\frac{5}{9}(C - 32^\circ) = \frac{9}{5}C + 32^\circ.$$

Solving,

$$C = F = -40^\circ.$$

f 4. The mass of the earth is 80 times that of the moon and the diameter of the earth is $3\frac{2}{3}$ times that of the moon. From these data the acceleration on the moon is found to be approximately $\frac{1}{8}$ that on the earth, and since the distance to which a person will rise is inversely proportional to the acceleration, the man if on the moon would jump 6 times 3 feet or 18 feet high.

f 5. Let x = the distance from the top of the tower, where the bullet overtakes the stone.

Let v = the velocity of the bullet at the moment the stone is dropped and

$$s = 27.$$

Then

$$v = \sqrt{2gs} = \sqrt{54g} = 3\sqrt{6g}$$

and

$$x = (3\sqrt{6g})t + \frac{1}{2}gt^2. \quad (1)$$

But

$$x - 45 = \frac{1}{2}gt^2.$$

$$\therefore x = (3\sqrt{6g})t + x - 45,$$

whence

$$(\sqrt{6g})t = 15 \text{ and } t^2 = \frac{225}{6g}.$$

$$\text{Then } \frac{1}{2}gt^2 = \left(\frac{1}{2}\right)(g)\left(\frac{225}{6g}\right) = 18\frac{3}{4} \text{ ft.} \quad (2)$$

Substituting in (1), we have

$$x = 45 + 18\frac{3}{4}, \text{ or } 63\frac{3}{4} \text{ ft.}$$

$\therefore$ the bullet overtakes the stone $63\frac{3}{4}$ ft. below the top of the tower, after having fallen $90\frac{3}{4}$ ft. and the stone has fallen $18\frac{3}{4}$ ft.

$$f 6. \quad \text{Kinetic Energy} = FS = \frac{Wv^2}{2g}$$

or

$$F = \frac{Wv^2}{2gS'}$$

where $S = \frac{1}{8}$ ft., $W = 2$ lb., and $v = 12$ ft.

Then

$$F = \frac{2(12)^2}{(2)(32.16)(\frac{1}{48})}$$

$$= \frac{2 \times 144 \times 48}{2 \times 32 \cdot 16}, \text{ or } 215.$$

 $\therefore$ the force = 215 pounds.f 7. Let x = the depth of the well. t = the time taken by the stone to fall the distance x . t' = time it took the sound to reach the observer, who was situated at a distance equal to the distance covered by the sound during one second, that is, equal to the speed of sound. v = the speed of sound = 340 meters.

From the law of falling bodies and the conditions stated we get the following equations:

$$x = \frac{gt^2}{2} \quad (1)$$

$$x + v = vt' \quad (2)$$

$$t = 2t' \quad (3)$$

Eliminating t and t' and solving for x , we get

$$x = \frac{v[(v - 4g) \pm \sqrt{v(v - 8g)}]}{4g}$$

Using $g = 9.8$ meters and $v = 340$ meters, we get
 $x = 22.55$ meters, or 5.195 meters.f 8. For bodies above the earth's surface the weight varies inversely as the square of the distance from the center. Hence, to weigh $\frac{1}{16}$ as much as at the surface, the body must be $\sqrt{16}$, or 4 times as far from the center. The required height above the surface is therefore 3×4000 miles, or 12,000 miles.

f 9. The latent heat of ice is .505, that of steam .48; heat of fusion of ice is 80 and heat of vaporization of water 537. Then

$$[\frac{5}{9}(32 - 14) \times .505 + 80 + \frac{5}{9} \text{ of } 180]100 = 18,505,$$
the heat units necessary to melt the ice and raise the temperature of the resulting water to 212° F.

$$[\frac{5}{9}(270 - 212) \times .48 + 537]80 = 44,197\frac{1}{3},$$
the heat units set free in reducing steam from 270° F. to water at 212°.

$$(44,197\frac{1}{3} - 18,505) \div 537 = 47.844 \text{ lbs. of steam at } 212^\circ \text{ F. remaining.}$$
Hence the result is 47.844 lbs. steam at 212° F. and 132.156 lbs. water at 212° F.
f 10. Let x = volume in cubic centimeters and y = the specific gravity of ice.Then, weight of ice = xy grams
and weight of water displaced = $(\frac{9}{10})x$ grams.

$$\therefore xy = \frac{9}{10}x \text{ and } y = \frac{9}{10}.$$

Since the block just floats in alcohol, sp. g. = $\frac{9}{10}$; i.e. the specific gravity of alcohol is $\frac{9}{10}$.f 11. Let V = the new volume. v = the old volume. T° = the new temperature. t° = the old temperature.

Then, the pressure being constant, we have

$$V = \frac{v(273^\circ + T^\circ)}{273^\circ + t^\circ}.$$

But $V = \frac{1}{2}v$ and $t^\circ = 27^\circ$.

$$\therefore \frac{1}{2} = \frac{273^\circ + T^\circ}{273^\circ + 27^\circ}.$$

$$546^\circ + 2T^\circ = 273^\circ + 27^\circ$$

$$\text{or } T = -123^\circ.$$

$$\text{Proof. } V = \frac{v(273^\circ - 123^\circ)}{273^\circ + 27^\circ} = \frac{150v}{300} = \frac{1}{2}v.$$

f 12. "We are told that the correct answer is that the weight will move up as fast as the monkey and that they will ultimately meet at the top. The monkey, therefore, does twice the work of lifting himself to that height. This is said to have been crudely confirmed by a boy who found it far more difficult to pull himself up in such a case than when the top of the rope was immovably secured.

"It seems necessary to distinguish between a jerky and a uniform movement of the monkey; the former involves acceleration, deceleration, and inertia. It is claimed that with a uniform motion the weight would not move, 'as the monkey cannot pull with a greater force than his weight. And that with a jerky upward motion of the monkey, involving acceleration and deceleration, the weight would move up and down for each jerk, but its average and ultimate position would remain the same. Others claim that the weight would move up with every jerk, but would not descend again during the deceleration, hence its ultimate upward motion would be equal to that of the monkey. A spring or elastic rope introduces another complication." From "Science."

f 13. The ball has to fall from a height of 257.28 ft. Whether the force of gravity draws it vertically downward as a freely falling body or acts with the projecting force to produce a curved path, it will reach the ground in the same time. The time required for it to fall is found by using the formula

$$l = \frac{1}{2}gt^2, \text{ where } l = 257.28 \text{ and } g = 32.16.$$

$$\therefore t = 4 \text{ seconds.}$$

If it would take 4 seconds to fall from the highest point reached, it would take 4 seconds for it to rise to that height and it would therefore be in the air 8 seconds. During each of these seconds it has a horizontal motion of 1000 ft. Its range is therefore 8000 ft.

f 14.

$$l = \frac{1}{2}gt^2.$$

Then,

$$787.92 = 16.08 t^2$$

or

$$t = 7.$$

The second body was 7 seconds in falling 787.92 ft. The first body was therefore 10 seconds in falling. It fell 100×16.08 ft., or 1608 ft.

f 15. It is a case of echo. It took the sound $1\frac{1}{4}$ seconds to reach the reflecting surface. If the velocity of sound at $0^\circ \text{C.} = 1090$ ft., the required distance $= 1\frac{1}{4} \times 1090$ ft., or 1362.5 ft.

f 16. The ice requires 8000 heat units. Each pound of steam will supply 637 heat units.

$\therefore 8000 \div 637 = 12.55^+$, the number of pounds required.

f 17. The weight of the bar, 4 lbs., may be considered as concentrated at the center of mass, 5 in. from either end.

Let x = the distance of the fulcrum from the end at which the 6-lb. weight is hung.

Then we have

$$6 \cdot x = 4(5 - x).$$

$$\therefore x = 2.$$

f 18. The observer would overtake in inverse order the pulses that started ahead of him. The music would be heard again as if played backward.

f 19. Its volume must be not less than that of 1 kg. of water. The volume of 1 kg. of water is 1 l., 1 cu. dm., or 1000 cu. cm.

f 20. (a) The density of a solid heavier than water is found by dividing its weight in air by the loss of weight in water. Thus,

$$D = \frac{w}{w - w'},$$

where w = weight in air and w' = weight of solid immersed in water.

Then

$$11.35 = \frac{600}{600 - w'}$$

$\therefore w' = 547.13$ gr., the weight of the lead in water.

(b) Weight of both in air	= 900 gr.
Weight of both in water	= 472.5 gr.
Weight lost by both in water	= 427.5 gr.
Weight lost by lead in water	= 52.87 gr.
Weight lost by wood in water	= 374.63 gr.

$\therefore$ density of wood = $\frac{300}{374.63}$, or .8+.

f 21. Since the intensity of a sound diminishes as the square of the distance from the source, the intensity of the sound made by the trio of noise makers six feet away may be represented by the number $\frac{3}{8}$ and that of the two at a distance of four feet by the number $\frac{2}{16}$. Thus it is seen that the disturbance made by the two youngsters would be $1\frac{1}{2}$ times as intense as that made by the three.

f 22. Let x = depth of the well in feet.

v = velocity of sound at 0° C.

Then by the law of falling bodies

$$x = \frac{1}{2}gt^2, \text{ or } t = \sqrt{\frac{2x}{g}}, \text{ time of falling.}$$

Also $\frac{x}{v} = \frac{x}{1090.5}$, time required for the sound to reach the top of the well.

$$\therefore \sqrt{\frac{2x}{32.16}} + \frac{x}{1090.5} = 5.$$

$$\therefore x = 351.8 \text{ ft.}$$

TRIGONOMETRY

g 1. Let A , $2A$, and $180 - 3A$ denote the $\angle$ s opposite the sides, x , $x + 1$, and $x + 2$ respectively.

Then by Law of Sines we have

$$\frac{\sin 2A}{\sin A} = \frac{2 \sin A \cos A}{\sin A} = \frac{x + 2}{x}.$$

$$\text{Hence, } \cos^2 A = \left(\frac{x + 2}{2x}\right)^2. \quad (1)$$

Also by Law of Sines,

$$\frac{\sin (180 - 3A)}{\sin A} = \frac{\sin 3A}{\sin A} = \frac{x + 1}{x}.$$

$$\text{Since } \sin 3A = 3 \sin A - x \sin^3 A,$$

$$\text{we get } 3 - 4 \sin^2 A = \frac{x + 1}{x}$$

$$\text{or } \sin^2 A = \frac{2x - 1}{4x}. \quad (2)$$

Adding (1) and (2), we have,

$$\frac{2x - 1}{4x} + \left(\frac{x + 2}{2x}\right)^2 = 1. \quad (3)$$

Solving for x in (3) gives $x = 4$.

$\therefore$ the sides of the $\triangle$ are 4, 5, and 6.

$$\begin{aligned} g 2. \quad \sin 3x &= \sin (2x + x) \\ &= \sin 2x \cos x + \cos 2x \sin x. \end{aligned}$$

Since $\sin 2x = 2 \sin x \cos x$ and

$$\cos 2x = \cos^2 x - \sin^2 x,$$

$$\begin{aligned} \therefore \sin 3x &= 2 \sin x \cos^2 x + \sin x \cos^2 x - \sin^3 x \\ &= 3 \sin x \cos^2 x - \sin^3 x. \end{aligned}$$

Since $\cos^2 x = 1 - \sin^2 x$,

$$\begin{aligned} \therefore \sin 3x &= 3 \sin x - 3 \sin^3 x - \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x. \end{aligned}$$

$$\begin{aligned} g 3. \quad \cos 3x &= \cos (2x + x) \\ &= \cos 2x \cos x - \sin 2x \sin x \\ &= (2 \cos^2 x - 1) \cos x - 2 \sin^2 x \cos x \\ &= 4 \cos^3 x - 3 \cos x. \end{aligned}$$

$$\begin{aligned}
 g \ 4. \quad \tan 3x &= \tan (2x + x) \\
 &= \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x} \\
 &= \frac{\frac{2 \tan x}{1 - \tan^2 x} + \tan x}{1 - \frac{2 \tan^2 x}{1 - \tan^2 x}} \\
 &= \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}
 \end{aligned}$$

g 5. Since $C = 180 - (A + B)$, we have $\sin C = \sin (A + B)$ and $\cos C = -\cos (A + B)$. Then

$$\begin{aligned}
 \sin 2A + \sin 2B &= 2 \sin (A + B) \cos (A - B) \\
 &= 2 \sin C \cos (A - B).
 \end{aligned}$$

$$\sin 2C = 2 \sin C \cos C = -2 \sin C \cos (A + B).$$

$$\begin{aligned}
 \therefore \sin 2A + \sin 2B + \sin 2C &= 2 \sin C [\cos (A - B) - \cos (A + B)] \\
 &= 2 \sin C \cdot 2 \sin A \sin B \\
 &= 4 \sin A \sin B \sin C.
 \end{aligned}$$

$$\begin{aligned}
 g \ 6. \quad \sin 75^\circ + \sin 15^\circ &= 2 \sin 45^\circ \cos 30^\circ \\
 \text{and} \quad \sin 75^\circ - \sin 15^\circ &= 2 \cos 45^\circ \sin 30^\circ. \\
 \therefore \frac{\sin 75^\circ + \sin 15^\circ}{\sin 75^\circ - \sin 15^\circ} &= \frac{2 \sin 45^\circ \cos 30^\circ}{2 \cos 45^\circ \sin 30^\circ} \\
 &= \tan 45^\circ \cot 30^\circ \\
 &= \tan 60^\circ.
 \end{aligned}$$

g 7. Given $B = 2C$ and $A = 2B = 4C$.

$$\therefore A + B + C = 7C = 180^\circ.$$

$$\therefore C = 25^\circ 42' 51\frac{3}{7}''.$$

$$B = 51^\circ 25' 42\frac{6}{7}''.$$

$$A = 102^\circ 51' 25\frac{5}{7}''.$$

Since

$$\frac{a}{c} = \frac{\sin A}{\sin C},$$

$$\begin{aligned}
 \therefore \log \frac{a}{c} &= \log \sin A + \text{colog} \sin C \\
 &= (9.98897 - 10) + 0.36263 \\
 &= 0.35160.
 \end{aligned}$$

$$\therefore \frac{a}{c} = 2.247, \text{ or } a = 2.247 c.$$

Also $\frac{b}{c} = \frac{\sin B}{\sin C}.$

$$\begin{aligned}
 \therefore \log \frac{b}{c} &= \log \sin B + \text{colog} \sin C \\
 &= (9.89311 - 10) + 0.36263 \\
 &= 0.25574.
 \end{aligned}$$

$$\therefore \frac{b}{c} = 1.802, \text{ or } b = 1.802 c.$$

$$\begin{aligned}
 \therefore a + b + c &= c(2.247 + 1.802 + 1) \\
 &= 5.049 c.
 \end{aligned}$$

$$\therefore c = \frac{100}{5.049} = 19.806.$$

$$a = 2.247 c = 44.504.$$

$$b = 1.802 c = 35.690.$$

g 8. Let x = length of pole,
 y = height of mound,
 a = horizontal distance of the observer.

Then,

$$y = a \tan 30^\circ,$$

$$y + x = a \tan 60^\circ,$$

$$\frac{y + x}{y} = \frac{\tan 60^\circ}{\tan 30^\circ} = 3.$$

$$\therefore y + x = 3y.$$

$$\therefore x = 2y.$$

h 4. $\rho = a \cos 2\theta$ (the four-leaved rose).

Since $\rho = 0$ when $\theta = \frac{\pi}{4}$, we see that if θ varies from 0 to $\frac{\pi}{4}$, the radius vector sweeps over the area OAB .

$\therefore$ area of one loop

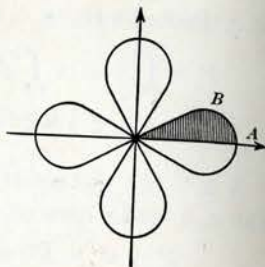
$$\begin{aligned} &= 2 \int_0^{\frac{\pi}{4}} \frac{1}{2} \rho^2 d\theta = \int_0^{\frac{\pi}{4}} a^2 \cos^2 2\theta d\theta \\ &= a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta = a^2 \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2} \cos 4\theta\right) d\theta \\ &= a^2 \left[\frac{1}{2}\theta + \frac{1}{8} \sin 4\theta + c\right]_0^{\frac{\pi}{4}} = \frac{\pi a^2}{8}. \end{aligned}$$

h 5. (a) Taking the logarithm of both sides, we have $\log y = \frac{1}{2} [\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4)]$.

Differentiating both sides with respect to x we have

$$\begin{aligned} \frac{1}{y} \frac{dy}{dx} &= \frac{1}{2} \left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} \right] \\ &= -\frac{2x^2 - 10x + 11}{(x-1)(x-2)(x-3)(x-4)} \\ \therefore \frac{dy}{dx} &= -\frac{2x^2 - 10x + 11}{(x-1)^{\frac{1}{2}}(x-2)^{\frac{1}{2}}(x-3)^{\frac{3}{2}}(x-4)^{\frac{3}{2}}} \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \frac{dy}{dx} &= \frac{\frac{d}{dx} \left(\frac{x^2+1}{x^2-1} \right)}{\frac{x^2+1}{x^2-1} \sqrt{\left(\frac{x^2+1}{x^2-1} \right)^2 - 1}} \\ &= \frac{\frac{(x^2-1)(2x) - (x^2+1)2x}{(x^2-1)^2}}{\frac{x^2+1}{x^2-1} \cdot \frac{2x}{x^2-1}} = -\frac{2}{x^2+1}. \end{aligned}$$



$$\begin{aligned} \text{(c)} \quad \frac{dy}{dx} \left[\log \frac{1+\sqrt{x}}{1-\sqrt{x}} \right] &= \frac{dy}{dx} [\log(1+\sqrt{x}) - \log(1-\sqrt{x})] \\ &= \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+\sqrt{x}} + \frac{\frac{1}{2}x^{-\frac{1}{2}}}{1-\sqrt{x}} \\ &= \frac{x^{-\frac{1}{2}}}{1-x}. \end{aligned}$$

h 6. Let $x = a \sin \theta$; then $dx = a \cos \theta d\theta$.
and $\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2 \theta)} = a \cos \theta$.

When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \frac{1}{2}\pi$.

Then

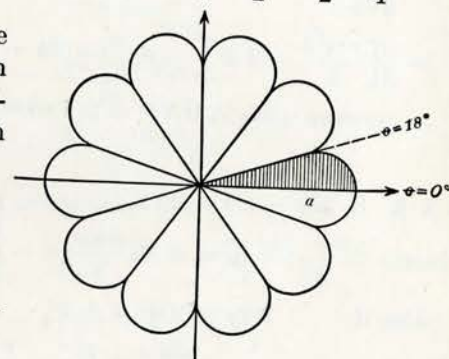
$$\begin{aligned} \int_a^0 \sqrt{a^2 - x^2} dx &= \int_0^{\frac{1}{2}\pi} a^2 \cos^2 \theta d\theta = \frac{1}{2} a^2 \int_0^{\frac{1}{2}\pi} (1 + \cos 2\theta) d\theta \\ &= \frac{1}{2} a^2 \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{1}{2}\pi} = \frac{1}{2} a^2 \cdot \frac{\pi}{2} = \frac{1}{4} \pi a^2. \end{aligned}$$

h 7. The curve $\rho = a \cos 5\theta$ when plotted will be a ten-leaved rose, as shown in the diagram.

Area of $\frac{1}{2}$ of a leaf

$$= \int_0^{\frac{\pi}{10}} \frac{1}{2} \rho^2 d\theta.$$

$\therefore$ area inclosed by curve



$$\begin{aligned} &= 20 \int_0^{\frac{\pi}{10}} \frac{1}{2} \rho^2 d\theta = 10 \int_0^{\frac{\pi}{10}} \rho^2 d\theta = 10 \int_0^{\frac{\pi}{10}} a^2 \cos^2 5\theta d\theta \\ &= 10 a^2 \int_0^{\frac{\pi}{10}} \left(\frac{1}{2} + \frac{1}{2} \cos 10\theta \right) d\theta \\ &= 10 a^2 \left[\frac{\theta}{2} + \frac{1}{20} \sin 10\theta + c \right]_0^{\frac{\pi}{10}} \\ &= 10 a^2 \left(\frac{\pi}{20} \right) = \frac{\pi a^2}{2}. \end{aligned}$$

h 8. Let MNP denote the equilateral triangle and $w = 62$ lbs., the weight of a cubic foot of water.

The equation of the line NP is

$$y - 0 = \frac{\sqrt{3}}{3}(x - 2\sqrt{3}).$$

$$\therefore y = \frac{x\sqrt{3} - 6}{3}.$$

Then the pressure upon $\triangle ONP$

$$\begin{aligned} &= w \int_{2\sqrt{3}}^0 \left(\frac{x\sqrt{3} - 6}{3} \right) x dx \\ &= \frac{w}{3} \int_{2\sqrt{3}}^0 (x^2\sqrt{3} - 6x) dx \\ &= \frac{w}{3} \left[\frac{x^3\sqrt{3}}{3} - 3x^2 \right]_{2\sqrt{3}}^0 = \frac{w}{3} [-(24 - 36)] = 4w. \end{aligned}$$

$$\therefore \text{pressure upon } \triangle MNP = 2(4w) = 496 \text{ lbs.}$$

h 9. If $x(y - e^x) = \sin x$,

then $y = e^x + \frac{\sin x}{x}.$

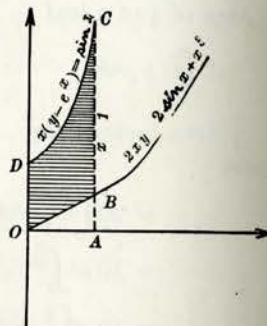
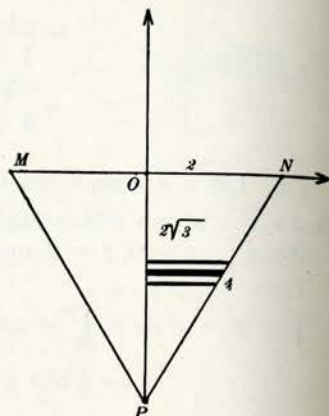
Also if $2xy = 2\sin x + x^3,$

then $y = \frac{\sin x}{x} + \frac{x^2}{2}.$

$$\text{Area of } OACD = \int_0^1 y dx$$

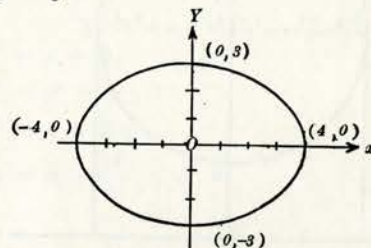
$$= \int_0^1 \left(e^x + \frac{\sin x}{x} \right) dx.$$

$$\text{Area of } OAB = \int_0^1 y dx = \int_0^1 \left(\frac{\sin x}{x} + \frac{x^2}{2} \right) dx.$$



$$\begin{aligned} \therefore \text{area of } OBCD &= \int_0^1 \left[\left(e^x + \frac{\sin x}{x} \right) - \left(\frac{\sin x}{x} + \frac{x^2}{2} \right) \right] dx \\ &= \int_0^1 \left(e^x - \frac{x^2}{2} \right) dx = \left[e^x - \frac{x^3}{6} \right]_0^1 \\ &= e - \frac{1}{6} - (1) = e - \frac{7}{6} = 1.55 + \dots \end{aligned}$$

h 10. If $\frac{x^2}{16} + \frac{y^2}{9} = 1,$



then

$$y^2 = \frac{144 - 9x^2}{16}$$

and

$$x^2 = \frac{144 - 16y^2}{9}.$$

Volume generated about OX .

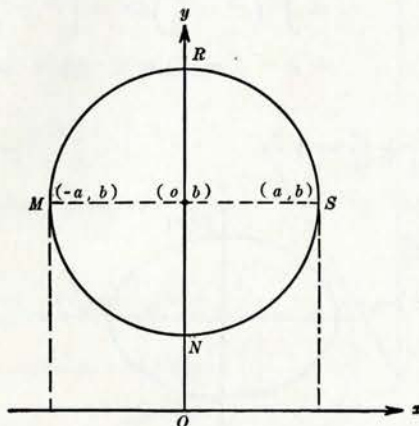
$$\begin{aligned} V_x &= \pi \int_{-4}^4 y^2 dx = \pi \int_{-4}^4 \frac{144 - 9x^2}{16} dx \\ &= \pi \left[9x - \frac{3x^3}{16} + c \right]_{-4}^4 \\ &= \pi [36 - 12 - (-36 + 12)] = 48\pi. \end{aligned}$$

Volume generated about OY .

$$\begin{aligned} V_y &= \pi \int_{-3}^3 x^2 dy = \pi \int_{-3}^3 \frac{144 - 16y^2}{9} dy \\ &= \pi \left[16y - \frac{16y^3}{27} + c \right]_{-3}^3 \\ &= \pi [48 - 16 - (-48 + 16)] = 64\pi. \end{aligned}$$

$$h \ 11. \quad x^2 + (y - b)^2 = a^2$$

$$\text{or} \quad (y - b)^2 = a^2 - x^2.$$



$$\therefore y - b = \pm \sqrt{a^2 - x^2}.$$

$$\therefore y = b \pm \sqrt{a^2 - x^2}.$$

$$\therefore \frac{dy}{dx} = \mp x(a^2 - x^2)^{-\frac{1}{2}}.$$

$y - b = +\sqrt{a^2 - x^2}$ is the equation of MRS , while

$y - b = -\sqrt{a^2 - x^2}$ is the equation of MNS .

Area generated by revolving MRS about OX

$$= S_x = 2\pi \int_{-a}^a (b + \sqrt{a^2 - x^2}) \left[1 + \frac{x^2}{a^2 - x^2}\right]^{\frac{1}{2}} dx.$$

Area generated by revolving MNS about OX

$$= S'_x = 2\pi \int_{-a}^a (b - \sqrt{a^2 - x^2}) \left[1 + \frac{x^2}{a^2 - x^2}\right]^{\frac{1}{2}} dx.$$

$$\therefore S_x + S'_x$$

$$= 2\pi \int_{-a}^a 2b \left[1 + \frac{x^2}{a^2 - x^2}\right]^{\frac{1}{2}} dx = 4\pi b \int_{-a}^a \left(\frac{a^2}{a^2 - x^2}\right)^{\frac{1}{2}} dx$$

$$= 4\pi ab \int_{-a}^a \frac{dx}{\sqrt{a^2 - x^2}} = \left[4\pi ab \left(\sin^{-1} \frac{x}{a}\right) + c\right]_{-a}^a$$

$$= 4\pi ab [\sin^{-1}(1) - \sin^{-1}(-1)]$$

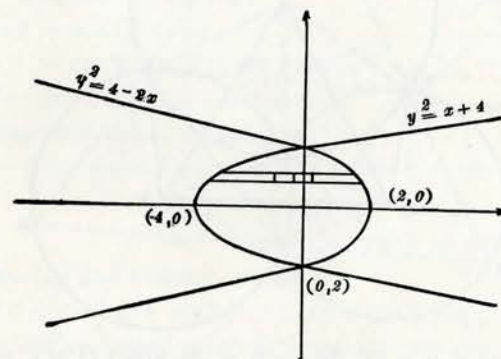
$$= 4\pi ab [2 \sin^{-1}(1)] = 4\pi ab \left[2 \cdot \frac{\pi}{2}\right] = 4\pi^2 ab.$$

$$h \ 12. \quad \text{If } y^2 = x + 4,$$

$$\text{then,} \quad x = y^2 - 4.$$

$$\text{If } y^2 = 4 - 2x,$$

$$\text{then } x = 2 - \frac{y^2}{2}.$$



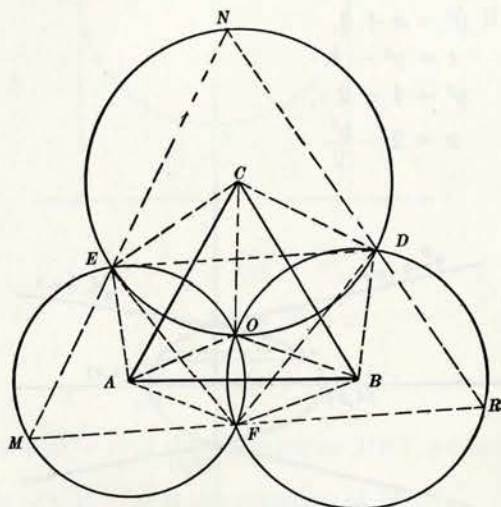
$$\begin{aligned} \therefore \frac{A}{2} &= \int_0^2 \int_{y^2-4}^{2-\frac{y^2}{2}} dx dy = \int_0^2 [x + c]_{y^2-4}^{2-\frac{y^2}{2}} dy \\ &= \int_0^2 \left[2 - \frac{y^2}{2} - y^2 + 4\right] dy = \int_0^2 \left[6 - \frac{3y^2}{2}\right] dy \\ &= \left[6y - \frac{y^3}{2} + c\right]_0^2 = 12 - 4 = 8. \end{aligned}$$

$$\therefore A = 2 \times 8 = 16.$$

NUTS FOR THE DOCTOR

1st Solution:

1. Let $\triangle ABC$ denote the garden and O the position of the tree.



Take $AO = a = 20$, $BO = b = 28$, and $CO = c = 31$.

Let $s =$ side of the garden.

Then $\text{area } \triangle ABC = \frac{1}{4} s^2 \sqrt{3}$.

• With B and C as centers and radii b and c describe circles. Draw EF , FD , and DE . $\angle EAF = 120^\circ$, being twice $\angle CAB$. Therefore $EF =$ side of an equilateral triangle inscribed in a circle of radius a .

Now $EF = a\sqrt{3}$. Likewise $FD = b\sqrt{3}$ and $ED = C\sqrt{3}$.

$$\text{Area } \triangle EAF = \frac{a^2}{4} \sqrt{3} = 100\sqrt{3} = 173.2051$$

$$\text{Area } \triangle BFD = \frac{b^2}{4} \sqrt{3} = 196\sqrt{3} = 339.4820$$

$$\text{Area } \triangle DCE = \frac{c^2}{4} \sqrt{3} = 240\frac{1}{4}\sqrt{3} = 416.1252$$

$$\text{Area } \triangle EFD = 823.1806$$

$$\therefore \text{area } AFBDC E \text{ or 2 times area } \triangle ABC = 1751.9929$$

$$\therefore \text{area } \triangle ABC = 875.996.$$

$$\text{But } \frac{1}{4} s^2 \sqrt{3} = 875.996.$$

$$\therefore s = 44.978 \text{ rods.}$$

2nd Solution:

From the figure, we have

$$\theta + \beta + \psi = 360^\circ,$$

$$\theta = 360^\circ - (\beta + \psi)$$

$$\text{and } \cos \theta = \cos (\beta + \psi)$$

$$= \cos \beta \cos \psi - \sin \beta \sin \psi,$$

$$\text{or } \cos \theta - \cos \beta \cos \psi = -\sin \beta \sin \psi.$$

Squaring both sides, we get

$$\begin{aligned} \cos^2 \theta - 2 \cos \theta \cos \beta \cos \psi + \cos^2 \beta \cos^2 \psi &= \sin^2 \beta \sin^2 \psi \\ &= (1 - \cos^2 \beta)(1 - \cos^2 \psi) \\ &= 1 - \cos^2 \beta - \cos^2 \psi + \cos^2 \beta \cos^2 \psi. \end{aligned}$$

Transposing and uniting, we have

$$\cos^2 \theta + \cos^2 \beta + \cos^2 \psi - 2 \cos \theta \cos \beta \cos \psi = 1. \quad (1)$$

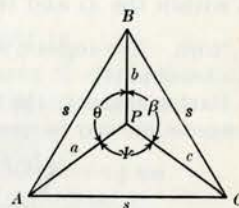
From the cosine formulas in trigonometry, we get

$$\cos \theta = \frac{a^2 + b^2 - s^2}{2ab} \quad (A), \quad \cos \beta = \frac{b^2 + c^2 - s^2}{2bc} \quad (B),$$

$$\cos \psi = \frac{c^2 + a^2 - s^2}{2ac} \quad (C).$$

Substituting these values in (1), clearing of fractions, and arranging terms, we get

$$s^6 - (a^2 + b^2 + c^2)s^4 + (a^4 + b^4 + c^4 - a^2c^2 - b^2a^2 - c^2b^2)s^2 = 0. \quad (2)$$



Factoring, we find one root is $s^2 = 0$. Removing the factor s^2 , we have

$$s^4 - (a^2 + b^2 + c^2)s^2 + a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 = 0. \quad (3)$$

This is a quadratic equation in s^2 . Solving equation (3) for s , we find

$$s = \pm \sqrt{\frac{a^2 + b^2 + c^2}{2}} \pm \frac{1}{2} \sqrt{6(a^2b^2 + b^2c^2 + c^2a^2) - 3(a^4 + b^4 + c^4)}. \quad (4)$$

Substituting values of a , b , and c , we have

$$s = \sqrt{\frac{2145}{2}} + \frac{1}{2} \sqrt{6(1451424) - 3(1698177)}.$$

$$\therefore s = 44.978 \text{ rd.}$$

The + signs before the radicals apply when the point P is within the Δ and the - signs when without.

NOTE. The angles θ , β , and ψ may be found by inserting the values of s found in (4).

Having found s , the following formulas, adapted for logarithmic computation, may be used to replace (A), (B), and (C).

$$\cos \frac{1}{2} \theta = \sqrt{\frac{p(p-s)}{ab}} \quad \text{in which} \quad p = \frac{1}{2}(a+b+s),$$

$$\cos \frac{1}{2} \beta = \sqrt{\frac{q(q-s)}{bc}} \quad \text{in which} \quad q = \frac{1}{2}(b+c+s),$$

$$\cos \frac{1}{2} \psi = \sqrt{\frac{r(r-s)}{ac}} \quad \text{in which} \quad r = \frac{1}{2}(a+c+s).$$

2. Let R denote the radius of the large circle and r the radius of the small circle.

$$\pi R^2 = 6,969,600 \text{ sq. ft.} \quad \therefore R = 1489.46 \text{ sq. ft.}$$

$$EF = \frac{R}{2} = 744.73 \text{ ft.}$$

$$\pi r^2 = 43,560 \text{ sq. ft.} \quad \therefore r = 117.752 \text{ ft.}$$

(a) To find $\angle FOK$.

$$\sin \frac{1}{2} \angle OEF = \frac{OF}{OE} = \frac{r}{R} = 8.89794.$$

$$\text{Then } \frac{1}{2} \angle OEF = 4^\circ 32' 3.7''.$$

$$\therefore \angle OEF = 9^\circ 4' 7.4''.$$

$$\angle EOF = \frac{180^\circ - (9^\circ 4' 7.4'')}{2}$$

$$= 85^\circ 27' 56.3''.$$

$$\angle EOJ = 240^\circ.$$

$$\angle FOK = \angle EOJ - 2 \angle EOF = 69^\circ 4' 7.4''$$

$$= 248,647.4''.$$

$$360^\circ = 1,296,000''.$$

(b) To find area sector FOK .

$$\frac{\text{Area sector } FOK}{43,560} = \frac{248647.4}{1296000}$$

$$\therefore \text{area sector } FOK = 8357.316 \text{ sq. ft.}$$

(c) To find area sector FEO .

$$\angle FEO = 9^\circ 4' 7.4'' = 32647.4''.$$

Area of circle with radius $FE = \frac{1}{4}$ area of circle of radius.

$$R = 1,742,400 \text{ sq. ft.}$$

$$\frac{\text{Area sector } FEO}{1742400} = \frac{32647.4}{1296000}$$

$$\therefore \text{area sector } FEO = 43,892.62 \text{ sq. ft.}$$

(d) To find area $\triangle OEF$.

$$\text{Area } \triangle OEF = \frac{1}{2} \overline{OE}^2 \sin \angle OEF.$$

$$\therefore \text{area } \triangle OEF = 43,709.54 \text{ sq. ft.}$$

$$\text{Area segment } OF = \text{sector } FEO - \triangle OEF$$

$$= 43,892.62 - 43,709.54, \text{ or } 183.08 \text{ sq. ft.}$$

$$\text{Segment } OF + \text{segment } OK = 366.16 \text{ sq. ft.}$$

$$\text{The pellicoid } FOK = \text{sector } FOK - 366.16$$

$$= 8357.316 - 366.16, \text{ or } 7991.156 \text{ sq. ft.}$$

(e) To find areas of A , P , and M .

$$\text{Area of sector } GEO = \frac{1}{6} \pi \overline{EO}^2 = \frac{1}{24} \pi \cdot R^2 = 290,400 \text{ sq. ft.}$$

$$\text{Area } \triangle GEO = \frac{\overline{EO}^2 \sqrt{3}}{4} = 240,158.62 \text{ sq. ft.}$$

$$\text{Area segment } OG = \text{sector } GEO - \triangle GEO = 50,241.38 \text{ sq. ft.}$$

$$\text{Area of } OFGF' = 2 \text{ area segment } OG = 100,482.76 \text{ sq. ft.}$$

Area A = area circle with center J passing through O
 $- (2 \text{ area } OFGF' + \text{area pellicoid } FOK)$
 $= 1,742,400 - (200,965.52 + 7991.156) \text{ or}$
 $1,533,443.324 \text{ sq. ft.}$

Area P = area $OFGF' - \frac{1}{3}(\pi r^2 - 3 \text{ pellicoids})$
 $= 100,482.76 - \frac{1}{3}(43,560 - 23,973.468)$
 $= 93,953.916 \text{ sq. ft.}$

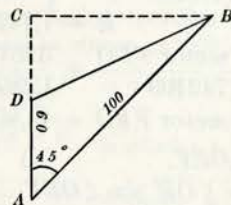
Area $M = \frac{1}{3}[\pi R^2 - (3A + 3P + \pi r^2)] = 681,282.76 \text{ sq. ft.}$

NOTE. The work may be checked by using the formula, $\pi R^2 = 3A + 3M + 3P + \pi r^2$.

3. Let A denote the starting point.

$AB = 100 \text{ mi.} = \text{distance } A \text{ goes northeast.}$

$AD = 60 \text{ mi.} = \text{distance } B \text{ goes north.}$



Draw $CB \perp$ to AD produced.

$\angle A = 45^\circ$ and $\angle C = 90^\circ$.

Then $\angle ABC = 45^\circ$ and $AC = CB$.

$$\overline{AC}^2 + \overline{CB}^2, \text{ or } 2\overline{AC}^2 = 100^2.$$

$$\therefore AC = CB = 50\sqrt{2}.$$

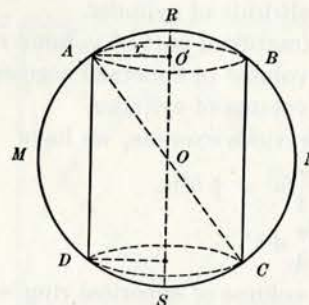
$$\therefore DB = \sqrt{(50\sqrt{2})^2 + (50\sqrt{2} - 60)^2} = 71.517.$$

Since they together travel 16 miles per hour, the time required for them to meet after turning toward each other $= \frac{1}{16}$ of 71.517, or 4.469 hours.

$\therefore$ distance A travels in 4.469 hours $= 144.69 \text{ mi.}$

$\therefore$ distance B travels in 4.469 hours $= 86.814 \text{ mi.}$

4. The required volume consists of the volume of a cylinder and the volume of two spherical segments.



Let RS = diameter of ball.

AB = diameter of auger hole.

V = volume of the cylinder.

V' = volume of one of the spherical segments.

Then $RS = D = 2R$ and $AB = d = 2r$.

$$OO' = \sqrt{AO'^2 - AO'^2} = \sqrt{R^2 - r^2}.$$

$AD = 2\sqrt{R^2 - r^2}$, the length of the cylinder.

$$\therefore V = \pi r^2 \cdot 2\sqrt{R^2 - r^2}.$$

Since $RO' = R - \sqrt{R^2 - r^2}$ and $AO' = r$,

$$\therefore 2V' = 2\left(\frac{1}{2}RO' \cdot \pi AO'^2 + \frac{1}{6}\pi RO'^3\right) \\ = \pi r^2(R - \sqrt{R^2 - r^2}) + \frac{1}{3}\pi(R - \sqrt{R^2 - r^2})^3.$$

$$\therefore V + 2V' = 2\pi r^2\sqrt{R^2 - r^2} + \pi r^2(R - \sqrt{R^2 - r^2}) \\ + \frac{1}{3}\pi(R - \sqrt{R^2 - r^2})^3 \\ = \pi r^2R + \pi r^2\sqrt{R^2 - r^2} + \frac{\pi}{3}(R - \sqrt{R^2 - r^2})^3 \\ = \pi[r^2R + r^2\sqrt{R^2 - r^2} + \frac{1}{3}(R - \sqrt{R^2 - r^2})^3].$$

5. Substituting in formula

$$V + 2V' = \pi\left[\frac{1}{4} + \frac{1}{4}\sqrt{1 - \frac{1}{4}} + \frac{1}{3}\left(1 - \sqrt{1 - \frac{1}{4}}\right)^3\right] \\ = \pi\left[\frac{1}{4} + \frac{1}{8}\sqrt{3} + \frac{1}{3}\left(\frac{1}{4} - \frac{1}{8}\sqrt{3}\right)\right] \\ = \frac{\pi}{6}[8 - 3\sqrt{3}], \text{ or } 1.4691 \text{ cu. ft.}$$

6. Let D = diameter of ball.
 d = diameter of cylinder.
 h = altitude of cylinder.
 m = fractional part of volume remaining.
 V = volume of spherical segment $ABNCMD$.
 V' = volume of cylinder.

Using figure in previous exercise, we have

$$V = \frac{\pi}{4}(d^2 + \frac{2}{3}h^2)h.$$

$$V' = \frac{\pi}{4}d^2h.$$

$$V - V' = \text{volume of spherical ring} = \frac{1}{8}\pi h^3.$$

$$\therefore \frac{1}{8}\pi h^3 = m(\frac{1}{8}\pi D^3).$$

$$\therefore h = Dm^{\frac{1}{3}}.$$

Since $DC = \sqrt{AC^2 - AD^2},$

$$\begin{aligned}\therefore d &= \sqrt{D^2 - D(\sqrt[3]{m})^2} = \sqrt{D^2 - D^2m^{\frac{2}{3}}} \\ &= D\sqrt{1 - m^{\frac{2}{3}}}.\end{aligned}$$

NOTE. Volume of spherical ring equals volume of a sphere whose diameter is height of the cylinder.

7. *Solution.* Using formula derived in the previous exercise, taking $D = 8$ and $m = \frac{1}{2}$.

$$\therefore d = 8\sqrt{1 - \frac{1}{2}\sqrt[3]{2}} = 4.8664 \text{ in.}$$

8. Using figure in Ex. 4

Here $AC = D = 10, AB = d = 8.$

$$O'O = \sqrt{5^2 - 4^2} = 3, AD = h = 6.$$

$$\text{Volume remaining} = \text{volume of spherical ring}$$

$$= \frac{1}{8}\pi h^3 \quad (\text{see Ex. 6})$$

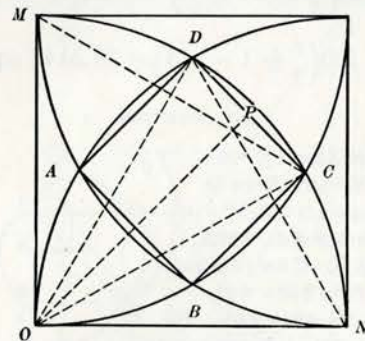
$$= \frac{1}{8}\pi \cdot 8^3.$$

$$\text{Volume of sphere} = \frac{1}{6}\pi d^3$$

$$= \frac{1}{6}\pi \cdot 10^3.$$

$$\therefore \text{part of sphere remaining} = \frac{\frac{1}{8}\pi \cdot 8^3}{\frac{1}{6}\pi \cdot 10^3} = .512.$$

9. *Solution.* Let the square MN denote the garden and $ABCD$ the area common to the four goats.



$$MO = OC = MC = OD = DN = ON = n.$$

$\triangle MOC$ and $\triangle DON$ are equilateral and $\angle DOC = 30^\circ$.

$$\text{Area of sector } DOC = \frac{1}{2} AD \cdot \text{arc } DC$$

$$= \frac{1}{2}[n(\frac{30}{360} \cdot 2\pi n)] = \frac{1}{12}\pi n^2.$$

$$DC = \sqrt{2n^2 - n\sqrt{4n^2 - n^2}} = n\sqrt{2 - \sqrt{3}}.$$

$$OP = \sqrt{n^2 - \left[\frac{n}{2}\sqrt{2 - \sqrt{3}}\right]^2} = \frac{n}{2}\sqrt{2 + \sqrt{3}}.$$

$$\text{Area of } \triangle DOC = \frac{1}{2} DC \cdot OP$$

$$= \frac{n}{2}\sqrt{2 - \sqrt{3}} \cdot \frac{n}{2}\sqrt{2 + \sqrt{3}}$$

$$= \frac{n^2}{4}.$$

$$\text{Area of segment } DC = \frac{1}{12}\pi n^2 - \frac{1}{4}n^2 = \frac{1}{12}n^2(\pi - 3).$$

$$\text{Area of square } ABCD = \overline{DC}^2 = n^2(2 - \sqrt{3}).$$

$$\therefore \text{area of } ABCD = n^2(2 - \sqrt{3}) + 4[\frac{1}{12}n^2(\pi - 3)].$$

10. The required area

$$A = n^2 \left(\frac{\pi}{3} + 1 - \sqrt{3} \right)$$

$$= 100 \left(\frac{\pi}{3} + 1 - \sqrt{3} \right) = 31.5147 \text{ sq. yd.}$$

1ST SOLUTION

11.

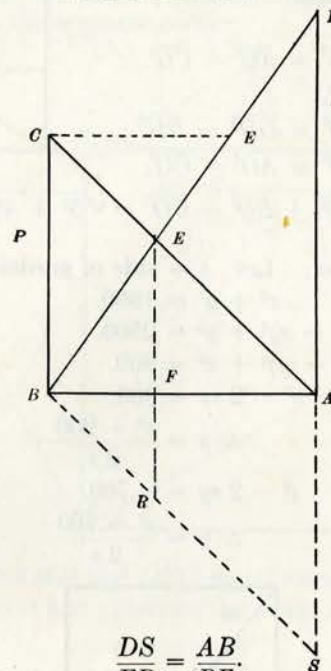
Let $AB = 25 \text{ ft.} = a$;
 $AC = AL = 4a^2$. Area to
 be grazed is $GQLAC +$
 $AEPH + GEP + PHL - ABCD$.
 $GQLAC = 12\pi a^2$; $PB = 3a$,
 $\text{and } PH^2 + BH^2 = 9a^2$.
 $PH^2 + (PH - a)^2 = 9a^2$; and
 $PH^2 - aPH = 4a^2$, $PH = \frac{1}{2}a(1 + \sqrt{17})$.
 $GL = 4a - PH = \frac{1}{2}a(7 - \sqrt{17})$.
 $GEP + PHL = 2GEP = \frac{1}{2}PE \times GE + GE^2/4$.
 The required area is
 $12\pi a^2 + \frac{1}{2}a^2(1 + \sqrt{17})^2 - a^2 + \frac{1}{2}a^2(1 + \sqrt{17})(7 - \sqrt{17})$
 $+ \frac{1}{16}a^2(7 - \sqrt{17})^2 \div (1 + \sqrt{17})$
 $= 30290.0769 \text{ sq. ft.}$

2ND SOLUTION

$AB = 25$; $AC = 25\sqrt{2}$; $OB = \frac{25}{2}\sqrt{2}$
 $BK = 100 - 25 = 75$;
 $PO = \sqrt{EP^2 - OB^2} = \frac{25}{2}\sqrt{34}$.
 $AP = \frac{25}{2}(\sqrt{34} + \sqrt{2})$;
 $AL = AP/\sqrt{2}$
 $= \frac{25}{2}(\sqrt{17} + 1)$.
 $\text{Area } ALPF = AL^2 = \frac{625}{2}\sqrt{9 + \sqrt{17}}$.
 $\text{Area } ABCD = 25^2 = 625$ and
 $\text{Area } BEPDC = \frac{625}{2}(9 + \sqrt{17} - 2) = 3476\frac{1}{2}$.
 $EK = 100 - AL = \frac{25}{2}(7 - \sqrt{17})$.
 $\text{Area of segments } PFH \text{ and } PEK$
 $= EK/4 \times PE = \frac{1}{4}(2PE \times EK) = 3252 \text{ sq. ft.}$
 $\text{Area } AHLK = \frac{1}{2} \pi \times 100^2 = 23562$ "
 $23562 + 3252 + 3476 = 30290 \text{ sq. ft.}$

From "The School Visitor."

12. Let AD and BC represent the poles and AC and BD the ropes. Draw BS parallel to AC , meeting AD produced at S . Then $CB = AS$ and $\triangle DBS$ and EBR are similar.



Then

$$\frac{DS}{ER} = \frac{AB}{BF}$$

Also $\triangle DBA$ and EBF are similar.

Then

$$\frac{AD}{EF} = \frac{AB}{BF}$$

$$\therefore \frac{DS}{ER} = \frac{AD}{EF}$$

Substituting values,

$$\frac{22 + 15}{15} = \frac{22}{EF}$$

$$\therefore EF = 8.919.$$

13. Let $ACBD$ denote the lot and O the spring.

Draw GH through O parallel to CB .

Then in $\triangle AOC$

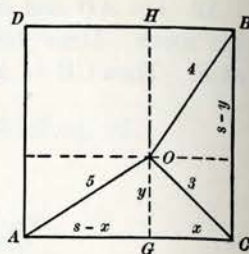
$$\overline{AO}^2 - \overline{CO}^2 = \overline{AG}^2 - \overline{CG}^2.$$

Also in $\triangle BOD$,

$$\overline{DO}^2 - \overline{BO}^2 = \overline{DH}^2 - \overline{BH}^2.$$

$$\therefore \overline{DO}^2 - \overline{BO}^2 = \overline{AO}^2 - \overline{CO}^2.$$

$$\therefore DO = \sqrt{\overline{AO}^2 + \overline{BO}^2 - \overline{CO}^2} = \sqrt{5^2 + 4^2 - 3^2}, \text{ or } 5.657.$$



14. 1st Solution: Let s = side of garden.

$$\text{Then } x^2 + y^2 = 1600. \quad (1)$$

$$(s-x)^2 + y^2 = 2500. \quad (2)$$

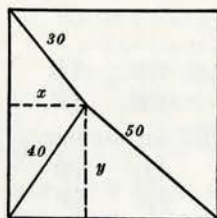
$$(s-y)^2 + x^2 = 900. \quad (3)$$

$$\text{Also } s^2 - 2sx = 900. \quad (2) - (1)$$

$$\therefore x = \frac{s^2 - 900}{2s}.$$

$$s^2 - 2sy = -700. \quad (3) - (1)$$

$$\therefore y = \frac{s^2 + 700}{2s}.$$



Since

$$x^2 + y^2 = 1600,$$

$$\therefore \left(\frac{s^2 - 900}{2s}\right)^2 + \left(\frac{s^2 + 700}{2s}\right)^2 = 1600$$

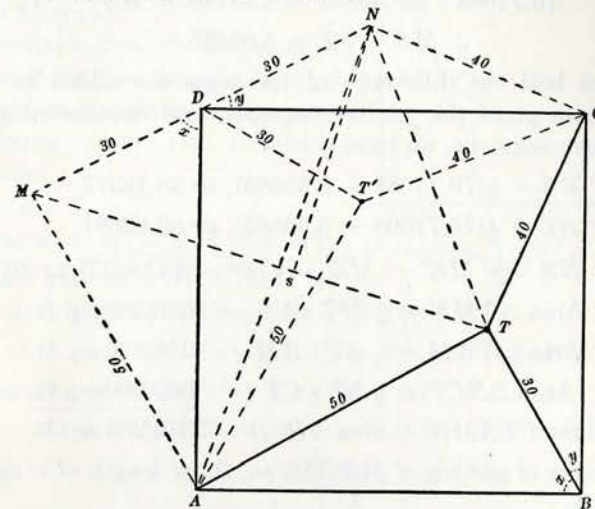
or

$$s^4 - 3400s^2 = -650,000.$$

$$\therefore s^2 = 3196.66 \text{ sq. ft., the area of the garden.}$$

2nd Solution:

Let T denote the tree and 30, 40, and 50 the distances to the three vertices. Let T' denote the position of the tree if placed near the opposite corner.



Construct $\triangle AMD$ and DNC equal respectively to $\triangle ATB$ and BTC . Let x and y denote the acute $\angle$ at B .

$$x + y = 90^\circ.$$

It is obvious that $\angle MDA$ and CDN are complementary.

Then $\angle MDA + \angle ADC + \angle CDN = 180^\circ$.

$\therefore MDN$ is a straight line.

$\angle MAT$ and TCN are rt. $\angle$.

$\therefore \triangle MAT$ and TCN are rt. $\triangle$.

Then

$$MS + ST = MT = \sqrt{50^2 + 50^2} = 50\sqrt{2} = 70.71068,$$

$$\text{and } TN = \sqrt{40^2 + 40^2} = 40\sqrt{2} = 56.56854.$$

Draw $SN \perp MT$.

Then in the scalene $\triangle MNT$, we have

$$* MT : MN + NT = MN - NT : MS - ST,$$

$$\text{or } 70.71068 : 60 + 56.56854 = 60 - 56.56854 : MS - ST.$$

$$70.71068 : 116.56854 = 3.43146 = MS - ST.$$

$$\therefore MS - ST = 5.65685.$$

Since half the difference of the segments added to half their sum gives the greater segment, and subtracted gives the lesser segment, we have

$$MS = \frac{1}{2}(70.71068 + 5.65685), \text{ or } 38.18377,$$

$$\text{and } ST = \frac{1}{2}(70.71068 - 5.65685), \text{ or } 32.02691.$$

$$\text{Then } NS = \sqrt{MN^2 - MS^2} = \sqrt{60^2 - 38.18377^2}, \text{ or } 46.281.$$

$$\text{Area } \triangle TMN = \frac{1}{2} MT \cdot NS = 1636.288 \text{ sq. ft.}$$

$$\text{Area } \triangle TAM = \frac{1}{2} AT \cdot AM = 1250.000 \text{ sq. ft.}$$

$$\text{Area } \triangle NCT = \frac{1}{2} NC \cdot CT = 800.000 \text{ sq. ft.}$$

$$\therefore \text{area } CTAMN = \text{area } ABCD = 3686.288 \text{ sq. ft.}$$

$$\therefore \text{area of garden} = 3686.288 \text{ sq. ft. or length of a side is } 60.69 \text{ ft.}$$

NOTE. The answer is not the same as in solution 1 because the distances to the corners are not taken in the same order.

3rd Solution:

Let x = side of the square garden with vertices A, C, D, B in counter-clockwise order. Let P denote the position of the tree such that $PA = a$, $PB = b$, $PC = c$, the given distances.

$$* \text{ Let } \overline{NS^2} = \overline{NT^2}.$$

$$\text{Then } \overline{MN^2} - \overline{MS^2} = \overline{NT^2} - \overline{ST^2}.$$

$$\therefore \overline{MS^2} - \overline{ST^2} = \overline{MN^2} - \overline{NT^2}.$$

$$\therefore (MS + ST)(MS - ST) = (MN + NT)(MN - NT).$$

$$\therefore MS + ST : MN + NT = MN - NT : MS - ST.$$

$$\therefore MT : MN + NT = MN - NT : MS - ST.$$

Rotate in the plane of the figure the $\triangle APB$ through 270° about A and thus obtain a quadrilateral $PAP'C$ with $\angle PAP' = 90^\circ$ and the diagonal $PP' = a\sqrt{2}$ while x denotes the other diagonal.

The area of a quadrilateral in terms of its four sides a, b, c, d and its diagonals m and n may be found from the following formula:

$$16 A^2 =$$

$$4 m^2 n^2 - (a^2 - b^2 + c^2 - d^2)^2.$$

Then area of $PAP'C$ with its sides a, a, b, c and its diagonals $a\sqrt{2}$ and x is

$$A = \frac{1}{4}[8 a^2 x^2 - b^4 - c^4 + 2 b^2 c^2]^{\frac{1}{2}}.$$

$$\text{Also area of } PAP'C = \triangle PAP' + \triangle P'CP.$$

$$\text{Area of } \triangle P'CP = [s(s - a\sqrt{2})(s - b)(s - c)]^{\frac{1}{2}},$$

where

$$s = \frac{(a\sqrt{2} + b + c)}{2}.$$

$$\text{Area of } \triangle PAP' = \frac{a^2}{2}.$$

$$\therefore \frac{1}{4}[8 a^2 x^2 - b^4 - c^4 + 2 b^2 c^2]^{\frac{1}{2}} = [s(s - a\sqrt{2})(s - b)(s - c)]^{\frac{1}{2}} \pm \frac{a^2}{2}$$

$$= \frac{1}{4}\{[b + c + a\sqrt{2}][b + c - a\sqrt{2}][a\sqrt{2} - (b - c)]$$

$$[a\sqrt{2} + (b - c)]\}^{\frac{1}{2}} \pm \frac{a^2}{2}$$

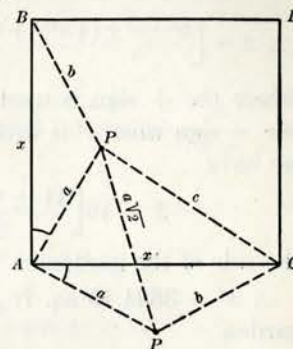
$$= \frac{1}{4}\{[2 bc + (b^2 + c^2 - 2 a^2)]$$

$$[2 bc - (b^2 + c^2 - 2 a^2)]\}^{\frac{1}{2}} \pm \frac{a^2}{2}$$

$$= \frac{1}{4}[2 b^2 c^2 - b^4 - c^4 - 4 a^4 + 4 a^2 b^2 + 4 a^2 c^2]^{\frac{1}{2}} \pm \frac{a^2}{2}.$$

$$8 a^2 x^2 = 4 a^2 b^2 + 4 a^2 c^2$$

$$\pm 4 a^2[4 a^2 b^2 + 4 a^2 c^2 + 2 b^2 c^2 - 4 a^4 - b^4 - c^4]^{\frac{1}{2}}.$$



$$\therefore x^2 = \frac{b^2 + c^2 \pm [4a^2b^2 + 4a^2c^2 + 2b^2c^2 - 4a^4 - b^4 - c^4]^{\frac{1}{2}}}{2}$$

$$\therefore x = \left[\frac{b^2 + c^2 \pm (4a^2b^2 + 4a^2c^2 + 2b^2c^2 - 4a^4 - b^4 - c^4)^{\frac{1}{2}}}{2} \right]^{\frac{1}{2}},$$

where the + sign is used when P is within the square and the - sign when P is without. For $a = 30$, $b = 40$, $c = 50$ we have

$$x = 10 \left[\frac{41 \pm \sqrt{1071}}{2} \right]^{\frac{1}{2}} = 60.7, \text{ or } 20.4,$$

the side of the garden.

$\therefore x^2 = 3684.49$ sq. ft., or 416.16 sq. ft., the area of the garden.

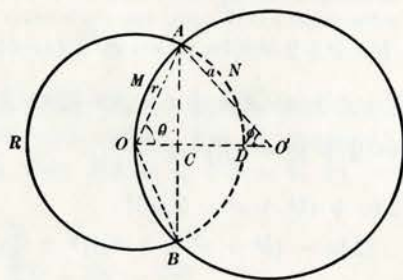
NOTE. The results in the two solutions do not agree because the distances a , b , and c were not taken in the same order as will be observed from the diagrams. For another solution, see the solution to Ex. 40, page 90.

15. Let O denote the stake to which the horse is tied and O' the center of the lake. Also let $\theta = \angle AOO'$, $\phi = \angle AO'O$.

$$r = AO = 100 \text{ ft. and } a = AO' = 117.75 \text{ ft.}$$

Then

$$a^2 = 13,865.5786.$$



$$\text{Since } \frac{r}{a} = \frac{\sin \phi}{\sin \theta}, \therefore \sin \theta = \frac{a}{r} \sin \phi. \quad (1)$$

$$\text{Since } r \cos \theta + a \cos \phi = a, \therefore \cos \theta = \frac{a}{r}(1 - \cos \phi). \quad (2)$$

Squaring and adding (1) and (2), we get

$$1 - \cos \phi = \frac{r^2}{2a^2}, \text{ or } 2 \sin^2 \frac{\phi}{2} = \frac{r^2}{2a^2}.$$

$$\therefore \sin \frac{\phi}{2} = \frac{r}{2a}. \quad (3)$$

$$\text{Since } \cos \theta = \frac{a}{r}(1 - \cos \phi) \text{ and } 1 - \cos \phi = \frac{r^2}{2a^2},$$

$$\therefore \cos \theta = \frac{r}{2a}. \quad (4)$$

$$\text{Area } \triangle AOC = \frac{r^2}{2} \sin \theta \cos \theta.$$

$$\text{Area } \triangle AO'C = \frac{a^2}{2} \sin \phi \cos \phi.$$

$$\text{Area sector } O-AND = \frac{r^2\theta}{2}.$$

$$\text{Area sector } O'-AMO = \frac{a^2\phi}{2}.$$

$$\text{Area segment } ANDB = r^2(\theta - \sin \theta \cos \theta).$$

$$\text{Area segment } AMOB = a^2(\phi - \sin \phi \cos \phi).$$

$$\therefore \text{area } AMOBDNA = r^2(\theta - \sin \theta \cos \theta) + a^2(\phi - \sin \phi \cos \phi).$$

$$\therefore \text{area } AGBOMA = \pi r^2 - r^2(\theta - \sin \theta \cos \theta) - a^2(\phi - \sin \phi \cos \phi).$$

Since r and a are known, $\sin \theta$, $\cos \theta$, $\sin \phi$, $\cos \phi$ are easily found from (3) and (4).

Solving for $\sin \theta$, $\cos \theta$, $\sin \phi$, $\cos \phi$, and substituting, we have

$$\text{area } AMOBDNA = 12,822.90 \text{ sq. ft.}$$

$$\text{and } \text{area } AGBOMA = 18,593.03 \text{ sq. ft.}$$

NOTE. If circle O' be a circular inclosure, and the horse be tied inside at O , the area grazed over is $12,822.9$ sq. ft.

16. The area grazed over is $ACMDBOA$. It consists of three parts: the semicircle CMD , and the triangular segments ACO and DBO . The curves CA and DB are arcs of the involute of circle O' .

Area ACO = area DBO

$$= \frac{1}{2} \int \rho^2 d\theta,$$

where ρ = the variable radius OC as it wraps around circle O' . But $OC = a\theta = \rho$.

Therefore $d\theta = \frac{d\rho}{a}$.

$$\therefore \text{area } ACO = \frac{1}{2} \int \rho^2 d\theta = \frac{1}{2a} \int_0^r \rho^2 d\rho = \frac{r^3}{6a}.$$

$$\therefore \text{area grazed over} = \frac{1}{2} \pi r^2 + \frac{r^3}{3a} = 18\,538.7669 \text{ sq. ft.}$$

17. Let $BC = a$, $AC = b$, and $EF = c$.

$\triangle ABC$ and FBE are similar and $\triangle FBE = \frac{1}{2}(\triangle ABC)$.

$$\therefore b^2 = 2c^2, \text{ or } b = c\sqrt{2}.$$

$$c = \frac{b}{\sqrt{2}} = .7071 b.$$

By the conditions, we have

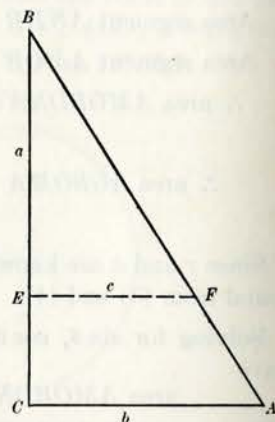
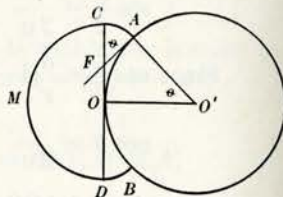
$$a + b = 70.$$

$$a - b = .7071 b.$$

$$\text{Solving, } a = 44.141,$$

$$b = 25.859,$$

$$c = 18.283.$$



18. Let x = the original number.

Then $\frac{x-1}{10}$ = original number with 1 detached.

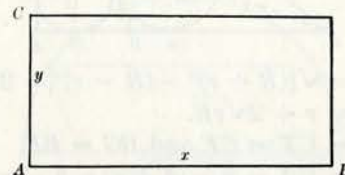
Transferring the 1 to the opposite end of a six-figure number, its value there is 100,000, hence the number is

$$\frac{x-1}{10} + 100,000 = \frac{x}{3}.$$

$$\therefore x = 42,8571.$$

19. Let x = no. balls in AB

and y = no. in AC .



Each additional course decreases the number in these rows by 1.

Let z = no. courses above the lowest.

Then $x - z = 11$, $y - z = 1$, $xy = 600$.

Solving, $x = 30$, $y = 20$, and $z = 19$.

There are then 20 layers, 11 in top, 24 in 2nd, 39 in 3rd, 56 in 4th, etc.

In this series the first difference $d_1 = 13$, 15, 17, etc.

Second difference $d_2 = 2$, 2, etc.

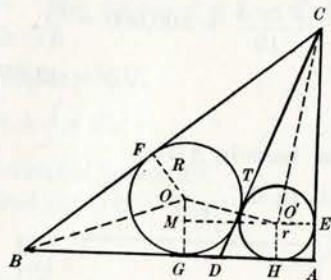
Third difference $d_3 = 0$, 0, etc.

$$\text{Sum of terms} = na + \frac{n(n-1)}{2} d_1 + \frac{n(n-1)(n-2)}{6} d_2,$$

where $a = 11$, first term, $n = 20$, no. of terms, $d_1 = 13$ and $d_2 = 2$.

$$\therefore S = 220 + \frac{20 \times 19}{2} \times 13 + \frac{20 \times 19 \times 18}{6} \times 2, \text{ or } 4970.$$

20. MO' is $\perp OG$, and since CD is $\perp OO'$,
 $OO' = R + r$; $OM = R - r$; $OG = GH$.



Then $OO' = \sqrt{[(R+r)^2 - (R-r)^2]} = 2\sqrt{rR}$. (1)
 $ME = r + 2\sqrt{rR}$. (2)

$CE = CT = CF$ and $BG = BF$.
 Also $ME = GA = 8 - (4 + r) = 4 - r$.

$\therefore r + 2\sqrt{rR} = 4 - r$.

$\therefore r + \sqrt{rR} = 2$.

$DH = DT = DG$. Hence $DH = \frac{1}{2}GH = \frac{1}{2}MO'$. (3)

Now $AD = DH + r = \frac{1}{2}(ME - r) + r = r + \sqrt{rR}$.

From (3) we find the numerical value of AD and have

$AD = r + \sqrt{rR} = 2$.

$CD = \sqrt{6^2 + 2^2} = 6.3246$.

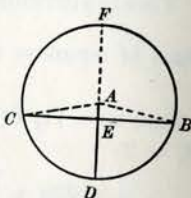
Dividing twice the area of each Δ by its perimeter, we find the radius of each inscribed circle

$r = 12 \div 14.3246$, or $.83772$.

$R = 36 \div 22.3246$, or 1.61258 .

21. Let A represent the angle at A of the right triangle BEA . Then, the area of the sector, BDC , is $25A$; area of the triangle, BAC , is $12\frac{1}{2} \sin 2A$; and area of segment, $E-BDC$, must be $25A - 12\frac{1}{2} \sin 2A$. Therefore,

$25A - 12\frac{1}{2} \sin 2A = 8\frac{1}{3} \text{ times } 3.1416$.

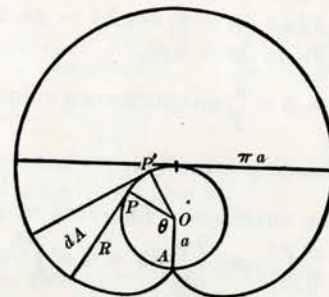


Dividing through by $12\frac{1}{2}$, we get

$2A - \sin 2A = 2.0944$.

Solving this equation by Double Position, we find that the angle A equals $74^\circ 38'$ and then the length of BC is $10 \sin A$, or 10 times .964225, which is 9.64225 feet.

22. Area grazed over by the cow consists of a semicircle and two equal portions of involutes of the given circle.



The area of one of the latter may be found by drawing tangents to the circle at nearby points such as P and P' . Let $\angle AOP = \theta$; then $\angle$ between tangents at P and P'
 $= \angle POP' = d\theta$.

Denoting area of portion of one involute by A , we have

$dA = \frac{1}{2} R^2 d\theta = \frac{1}{2} a^2 \theta^2 d\theta$

for

$R^2 = \text{arc } AP = a\theta$.

Here θ varies from 0 to π .

$\therefore A = \frac{1}{2} a^2 \int_0^\pi \theta^2 d\theta = \frac{1}{2} a \left[\frac{\theta^3}{3} + c \right]^\pi = \frac{1}{2} a^2 \pi^3$.

$\therefore$ total area grazed over

$= 2\left(\frac{1}{8} a^2 \pi^3\right) + \frac{\pi}{2} (\pi a^2)^2 = \frac{5}{8} \pi^3 a^2$.

23.* Let ϕ be the angle between the polar axis and the radius vector of the circle described on the radius vector of the lemniscate as a diameter. Let r be the radius vector of the circle, the origin and polar axis being the same as for the lemniscate. Then the equation of the circle is $r = \rho \cos(\phi - \theta)$. Or $r^2 = \rho^2 \cos^2(\phi - \theta) = a^2 \cos 2\theta \cos^2(\phi - \theta)$, substituting from the equation of the lemniscate. In this equation, θ is the variable parameter. Taking the derivative of this equation with respect to θ , we have

$$0 = 2a^2 \cos(\phi - \theta)[\sin(\phi - \theta) \cos 2\theta - \sin 2\theta \cos(\phi - \theta)] \\ = 2a^2 \cos(\phi - \theta) \sin(\phi - 3\theta).$$

Hence, $\theta = \frac{\phi}{3}$, or $\theta = \phi - \frac{\pi}{2}$, an extraneous value. Substituting

$\frac{\phi}{3}$ for θ in the preceding equation, we have $r^2 = a^2 \cos^3 \frac{2\phi}{3}$, the equation of the envelope. The area of one loop of the envelope is $A = \frac{a^2}{2} \int_0^a \cos^3 \frac{2\phi}{3} d\phi = 3a^2 \int_0^{\pi} \cos^3 2\theta d\theta = a^2$.

The area of one loop of the lemniscate is

$$A' = a^2 \int_0^{\frac{\pi}{4}} \cos 2\theta d\theta = \frac{a^2}{2}.$$

Hence,

$$A = 2A'.$$

$$24. \text{ Here } \frac{d\rho}{d\theta} = 2a[\cos \theta \tan \theta + \sin \theta \sec^2 \theta]$$

$$= 2a[\cos \theta \tan \theta + \tan \theta \sec \theta]$$

$$= 2a \tan \theta (\cos \theta + \sec \theta).$$

$$S = \int_0^{\frac{\pi}{4}} \left[\rho^2 + \left(\frac{d\rho}{d\theta} \right)^2 \right]^{\frac{1}{2}} d\theta \\ = \int_0^{\frac{\pi}{4}} [4a^2 \tan^2 \theta \sin^2 \theta + 4a^2 \tan^2 \theta (\cos \theta + \sec \theta)^2]^{\frac{1}{2}} d\theta \\ = \int_0^{\frac{\pi}{4}} [2a \tan \theta (\sin^2 \theta + \cos^2 \theta) + 2 + \sec^2 \theta]^{\frac{1}{2}} d\theta$$

* From *American Mathematical Monthly*.

$$S = \int_0^{\frac{\pi}{4}} 2a \tan \theta (3 + \sec^2 \theta)^{\frac{1}{2}} d\theta \\ = 2a \int_0^{\frac{\pi}{4}} \tan \theta (4 + \tan^2 \theta)^{\frac{1}{2}} d\theta \\ = 2a \int_0^{\frac{\pi}{4}} \tan \theta \sqrt{4 + \tan^2 \theta} d\theta.$$

$$\text{Let } r = \sqrt{4 + \tan^2 \theta}. \text{ Then } d\theta = \frac{r dr}{\tan \theta \sec^2 \theta}.$$

$$\text{When } \theta = \frac{\pi}{4}, r = \sqrt{5} \text{ and when } \theta = 0, r = 2.$$

$$= 2a \int_2^{\sqrt{5}} \tan \theta(r) \frac{r dr}{\tan \theta \sec^2 \theta} = 2a \int_2^{\sqrt{5}} \frac{r^2 dr}{\sec^2 \theta} = 2a \int_2^{\sqrt{5}} \frac{r^2 dr}{r^2 - 3} \\ = 2a \int_2^{\sqrt{5}} \left[1 + \frac{3}{r^2 - 3} \right] dr \\ = 2a \left[r + \frac{3}{2\sqrt{3}} \log \frac{r - \sqrt{3}}{r + \sqrt{3}} + c \right]_2^{\sqrt{5}} \\ = 2a \left[\sqrt{5} + \frac{\sqrt{3}}{2} \log \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} \right] - 2a \left[2 + \frac{\sqrt{3}}{2} \log \frac{2 - \sqrt{3}}{2 + \sqrt{3}} \right] \\ = 2a\sqrt{5} - 4a + a\sqrt{3} \left[\log \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} + \sqrt{3}} - \log \frac{2 - \sqrt{3}}{2 + \sqrt{3}} \right] \\ = 2a\sqrt{5} - 4a + a\sqrt{3} \left[\log \frac{5 - 3}{(\sqrt{5} + \sqrt{3})^2} - \log \frac{4 - 3}{(2 + \sqrt{3})^2} \right] \\ = 2a\sqrt{5} - 4a \\ + a\sqrt{3} [\log 2 - 2 \log (\sqrt{5} + \sqrt{3}) - 0 + 2 \log (2 + \sqrt{3})] \\ = 2a\sqrt{5} - 4a \\ - 2a\sqrt{3} [\log (\sqrt{5} + \sqrt{3}) - \log (2 + \sqrt{3}) - \frac{1}{2} \log 2] \\ = 2a\sqrt{5} - 4a - 2a\sqrt{3} \left[\log \frac{\sqrt{3} + \sqrt{5}}{2 + \sqrt{3}} - \log \sqrt{2} \right] \\ = 2a \left[\sqrt{5} - 2 - \sqrt{3} \left(\log \frac{\sqrt{3} + \sqrt{5}}{(2 + \sqrt{3})\sqrt{2}} \right) \right].$$

25.* The central equation of the elliptic field is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Let r_1 be the length of rope by which one cow is tethered at the point A_1 , the right-hand extremity of the major axis. The equation of the circle over which this cow can browse is $(x - a)^2 + y^2 = r^2$. The coördinates of the point of intersection, P_1 , of this circle with the ellipse are

$$(x_1, y_1) = \left(a \frac{[a^2 - \sqrt{b^4 + r^2(a^2 - b^2)}]}{(a^2 - b^2)}, \sqrt{2a^4 \frac{\sqrt{b^4 + r^2(a^2 - b^2)} - b^2(a^2 - b^2)r^2 - a^2(a^4 + b^4)}{(a^2 - b^2)}} \right).$$

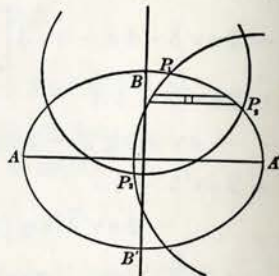
The area common to the ellipse and the circle is

$$\begin{aligned} \text{area} &= 2 \int_0^{y_1} \int_{a - \sqrt{r^2 - y^2}}^{\left(\frac{a}{b}\right)\sqrt{b^2 - y^2}} dx dy \\ &= 2 \left[\frac{a}{b} \frac{y_1}{2} \sqrt{b^2 - y_1^2} + \frac{b^2}{2} \sin^{-1} \frac{y_1}{b} - ay_1' + \frac{y_1'}{2} \sqrt{r^2 - y_1^2} + \frac{r^2}{2} \sin^{-1} \frac{y_1}{r} \right], \end{aligned}$$

which, by the conditions of the problem, $= \frac{1}{2} \pi ab$.

Substituting the value of y_1 in this equation, we have an equation in r . At the cost of great labor, this equation can be solved, by trial, to any degree of accuracy.

In a similar manner, the area over which a cow tethered at B can graze may be found. Let the coördinates of the points of intersection, P_2 , be (x_2, y_2) . The length of the radius BP_2 may be found to any desired degree of accuracy, as above. When this is found, x_2 and y_2 become known.



* This solution by Dr. B. F. Finkel, Drury College, was taken from the *American Mathematical Monthly*.

Then to find the area common to the two circles and the ellipse we have only to perform the following integration:

$$\int_{y^2}^{y^2} \int_{a - \sqrt{r^2 - y^2}}^{\left(\frac{a}{b}\right)\sqrt{b^2 - y^2}} dx dy + \int_{y_2}^{y_1} \int_{a - \sqrt{r^2 - y^2}}^{\left(\frac{a}{b}\right)\sqrt{b^2 - y^2}} dx dy,$$

where y^3 is the ordinate of P_3 .

The case when the circle whose center is B cuts the ellipse in four points would require still more labor.

26.* From the given equation, we have

$$x = \frac{9}{4} \sqrt{16 - y^2}; \quad dx = -\frac{9}{4} \frac{y dy}{\sqrt{16 - y^2}};$$

$$\text{and} \quad ds = \frac{1}{4} \sqrt{\frac{256 + 65 y^2}{16 - y^2}} \cdot dy.$$

Then the area of the surface generated is

$$\begin{aligned} 2\pi \int_{y=-4}^{y=4} x \cdot ds &= 2\pi \int_{-4}^4 \left(\frac{9}{4} \sqrt{16 - y^2} \right) \left(\frac{1}{4} \sqrt{\frac{256 + 65 y^2}{16 - y^2}} \right) dy \\ &= \frac{9\pi}{8} \int_{-4}^4 \sqrt{256 + 65 y^2} \cdot dy \\ &= \frac{9\pi\sqrt{65}}{16} \left[y \cdot \sqrt{y^2 + \frac{256}{65}} + \frac{256}{65} \log \left(y + \sqrt{y^2 + \frac{256}{65}} \right) \right]_{-4}^4 \\ &= \frac{9\pi\sqrt{65}}{16} \left\{ \frac{144}{\sqrt{65}} + \frac{256}{65} \log \left(4 + \frac{36}{\sqrt{65}} \right) \right. \\ &\quad \left. + \frac{144}{\sqrt{65}} - \frac{256}{65} \log \left(-4 + \frac{36}{\sqrt{65}} \right) \right\} \\ &= 162\pi + \frac{144\pi}{\sqrt{65}} \log \frac{9 + \sqrt{65}}{9 - \sqrt{65}}. \end{aligned}$$

27.* If we change these equations to parametric forms, we obtain

$$\begin{aligned} (1) \quad x &= 4a \cos^3 t \sin t, & y &= 2a \cos^2 t, \\ (2) \quad x &= a \sin t, & y &= ma^2 \sin^2 t + a \cos t, \\ (3) \quad x &= a \sin t, & y &= \frac{a^2 \cos t - a^2 b \sin t - c \csc t}{a}. \end{aligned}$$

* From *American Mathematical Monthly*.

Hence,

$$(1) \text{ Area} = \int y \, dx = \int_0^\pi 8a^2(4\cos^6 t - 3\cos^4 t) dt = \pi a^2;$$

$$(2) \text{ Area} = \int y \, dx = \int_0^{2\pi} (ma^2 \sin^2 t + a \cos t)a \cos t \, dt \\ = \pi a^2; \text{ and}$$

$$(3) \text{ Area} = \int y \, dx = \int_0^{2\pi} (a^2 \cos t - a^2 b \sin t - c \csc t) \cos t \, dt \\ = \pi a^2.$$

28.* We have the known definite integral

$$\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}.$$

Now let $z = \left(x - \frac{a}{x}\right)$. Then $dz = \left(1 + \frac{a}{x^2}\right)dx$,

and the integral becomes

$$\int_0^{\infty} e^{-(x-\frac{a}{x})^2} \left(1 + \frac{a}{x^2}\right) dx = \sqrt{\pi}$$

or

$$\int_0^{\infty} e^{-(x-\frac{a}{x})^2} dx + a \int_0^{\infty} e^{-(x-\frac{a}{x})^2} \frac{dx}{x^2} = \sqrt{\pi}.$$

In the second integral, let $x = \frac{a}{y}$. Then $dx = -\frac{a}{y^2} dy = \frac{a}{y^2} dy$,

and the second integral becomes

$$\int_0^{\infty} e^{-(\frac{a}{y}-y)^2} dy = \int_0^{\infty} e^{-(y-\frac{a}{y})^2} dy.$$

$$\text{Hence, } \int_0^{\infty} e^{-(x-\frac{a}{x})^2} dx + \int_0^{\infty} e^{-(y-\frac{a}{y})^2} dy = \sqrt{\pi}$$

or

$$2 \int_0^{\infty} e^{-(x-\frac{a}{x})^2} dx = \sqrt{\pi}$$

or

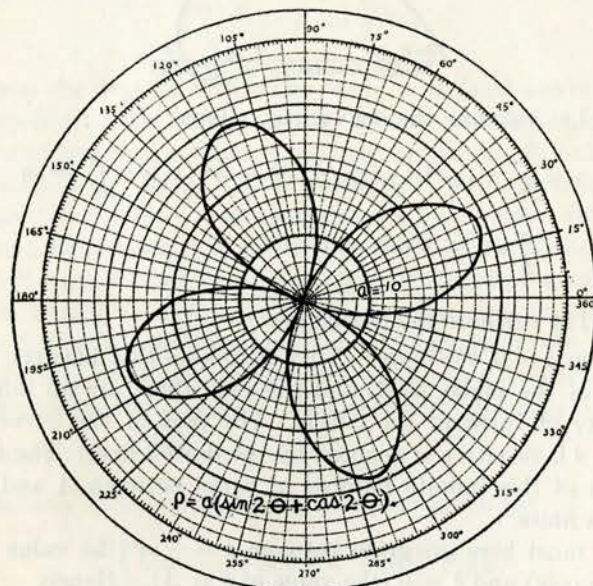
$$2 \int_0^{\infty} e^{-x^2 - \frac{a^2}{x^2} + 2a} dx = 2e^{2a} \int_0^{\infty} e^{-x^2 - \frac{a^2}{x^2}} dx = \sqrt{\pi}.$$

* From *American Mathematical Monthly*.

Whence,

$$\int_0^{\infty} e^{-(x^2 - \frac{a^2}{x^2})} dx = \frac{\sqrt{\pi}}{2e^{2a}}.$$

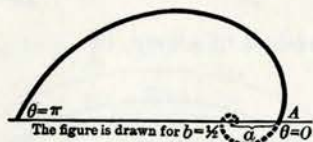
29. Area of one-half of a loop, or



$$\begin{aligned} \frac{A}{8} &= \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{1}{2} \rho^2 d\theta = \frac{1}{2} \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} a^2 (\sin 2\theta + \cos 2\theta)^2 d\theta \\ &= \frac{a^2}{2} \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (\sin^2 2\theta + 2 \sin 2\theta \cos 2\theta + \cos^2 2\theta) d\theta \\ &= \frac{a^2}{2} \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} (\sin 4\theta + 1) d\theta = \frac{a^2}{2} \left[-\frac{\cos 4\theta}{4} + \theta + c \right]_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \\ &= \frac{a^2}{2} \left[\left(0 + \frac{3}{8}\pi\right) - \left(0 + \frac{\pi}{8}\right) \right] = \frac{\pi a^2}{8}. \end{aligned}$$

$$\therefore A = 8 \times \frac{\pi a^2}{8} = \pi a^2.$$

30. Here $\frac{d\rho}{d\theta} = abe^{b\theta}$.



(a) Let s denote the heavy line. Then

$$s = \int_0^\pi \sqrt{(ae^{b\theta})^2 + (abe^{b\theta})^2} d\theta = a\sqrt{1+b^2} \int_0^\pi e^{b\theta} d\theta$$

$$= \frac{a}{b} \sqrt{1+b^2} (e^{b\pi} - 1).$$

(b) Let s denote the dotted line.

When $\rho = 0$, $\theta = -\infty$. Then it follows that between any point of the plane, as A , and the pole there are an infinite number of turns of the curve. The part of the curve for which θ is negative is denoted by the dotted line. The total length of this infinite number of turns between A and the pole is finite.

We must here integrate between $\theta = -\infty$ (the value of θ at the pole) and $\theta = 0$ (the value of θ at A). Hence,

$$s = a\sqrt{1+b^2} \int_{-\infty}^0 e^{b\theta} d\theta = \left[\frac{a}{b} \sqrt{1+b^2} (e^{b\theta}) \right]_{-\infty}^0 = \frac{a}{b} \sqrt{1+b^2}.$$

31. The curve of the type here given is generated by causing a circle of radius a to roll on a straight line. Then, a point on a fixed radius of this circle, and at a distance $\frac{\pi a}{3}$ from the center, will describe the given trochoid. The axis of X is the line on which the circle rolls, and the axis of Y is taken as the position of the radius containing the generating point when it is pointing vertically downward.

The desired area may be found by evaluating $\int y dx$ between the proper limits of integration. It is convenient to express this integral in terms of the parameter ϕ , as

$$\int \frac{a^2}{9} (3 - \pi \cos \phi)^2 d\phi.$$

From the way in which the curve is defined above, it is symmetrical with respect to the Y -axis; therefore, it will suffice to find that portion of the loop to the left of the Y -axis and multiply it by 2. As ϕ starts from zero and increases in value, the point which generates the curve starts from its position on the negative portion of the Y -axis and traverses the left half of the loop in a clock-wise direction about the origin, finally returning to the Y -axis again, this time in the upper half of the coördinate plane. The limits of integration in ϕ must, therefore, correspond to the two points where the loop cuts the Y -axis. Examination of the equations of the curve shows these to be 0 and $\frac{\pi}{6}$.

The area of half the curve is therefore given by

$$\frac{a^2}{9} \int_0^{\pi/6} (9 - 6\pi \cos \phi + \pi^2 \cos^2 \phi) d\phi$$

or
$$\frac{a^2}{9} \left(9\phi - 6\pi \sin \phi + \frac{\pi^2 \phi}{2} + \frac{\pi^2}{4} \sin 2\phi \right) \Big|_0^{\pi/6}.$$

$\therefore$ the area of the entire loop will then be double this, or

$$\frac{a^2 \pi}{108} (-36 + 2\pi^2 + 3\pi\sqrt{3}).$$

32.* *Solution I.* This problem is equivalent to the construction of $R\sqrt{5}$, where R is the radius of the given circle.

* From *American Mathematical Monthly*.

Let A be the center of the given circle. (Fig. 1.) Take any point B on its circumference and construct a circle of radius equal to that of the given circle. Let this circle intersect the given circle in C and D . Now with C as center

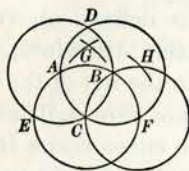


FIG. 1

construct another circle of radius R , intersecting the first two circles in E and F respectively. And finally with F as center construct another equal circle.

It is evident that $CD = R\sqrt{3}$. With E and F as centers and CD as radius strike arcs, intersecting in G . Then $CG = R\sqrt{2}$. With C as center and CG as radius strike an arc intersecting in H the circumference of the circle last drawn.

Then $EH = R\sqrt{5}$.

This also solves another problem due to Napoleon I; the quadrisection of the circumference of a circle by the use of the compasses alone, which is equivalent to the construction of $R\sqrt{2}$.

Solution II. Let AC be the given circle, of radius r . (Fig. 2.) From an arbitrary point A on its circumference, lay off three chords of radius length, reaching point H ; HA is then a diameter of AC . With H as a center and radius r lay off another circle finding its diameter CI in the same manner as before. Continue this until five such circles are drawn with their centers in the line AB . With H as a

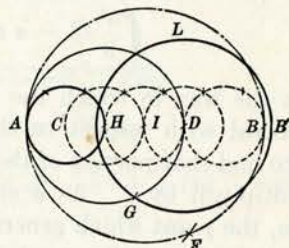


FIG. 2

center and $2r$ as a radius lay off the circle AD . With I as center and $3r$ as radius lay off a circle AB . With a radius of length $5r$ cut this circle in F , using A as center. Describe an arc with radius BF from A as a center, cutting circle AD in G . With DG as radius and D as center describe the circle LGB' which is a circle whose area is five times the area of AC . For chord $AF = 5r$, and diameter $AB = 6r$. Hence, chord $BF = \sqrt{36r^2 - 25r^2} = r\sqrt{11}$, chord $AG =$ chord $BF = r\sqrt{11}$, and diameter $AD = 4r$.

Hence, $DG = \sqrt{16r^2 - r^2} = r\sqrt{5}$. That is, circle LGB' is 5 times the given circle.

33. Let ABC be the triangle and P the given point.

Join A and P .

Construct $\angle CAE$ (without the triangle) equal to $\angle PAB$, and cut off AE so that $AP \cdot AE = \frac{1}{2} AB \cdot AC$.

On PE construct a segment of a circle containing an angle equal to the supplement of $\angle PAN$, or $\angle EAB$.

Let this segment cut AC in the point N , and let PN cut AB in L . The line PN bisects the triangle.

Proof. Since $LAEN$ is a cyclic quadrilateral

$$\angle ALP = \angle AEN.$$

Hence, $\triangle PAL$ and NAE are similar.

$$\text{Hence, } \frac{PA}{AL} = \frac{AN}{AE}$$

$$\text{or, } AN \cdot AL = PA \cdot AE = \frac{1}{2} AB \cdot AC.$$

Hence, PN bisects the triangle ABC .

This solution was contributed by J. Travers, B.A.B. Sc., M.R.S.T., Headmaster, Peterborough College, Harrow (England).

34.* We will assume relations expressed by (1), (2), (3).

$$r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}. \quad (1)$$

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \dots (2) \quad \Delta = \sqrt{rr_1r_2r_3}. \quad (3)$$

Since r_1, r_2, r_3 are given, r may be constructed as follows: Let A, B, C , be three points on a straight line. On same side of this line erect $\perp AD, BE, CF$, equal, respectively, to r_1, r_2, r_3 .

From G , intersection of AE and DB , drop $GH \perp$ to AC .

From K , intersection of GC and HF , drop $KL \perp$ to AC .

It is easily proved that

$$\frac{1}{GH} = \frac{1}{r_1} + \frac{1}{r_2}, \text{ and } \frac{1}{KL} = \frac{1}{GH} + \frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}.$$

$\therefore KL = r$, radius of inscribed circle.

From (2) and (1),

$$\frac{1}{r} - \frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{a}{\Delta}. \quad (4)$$

$$\therefore a = \Delta \left(\frac{1}{r_2} + \frac{1}{r_3} \right) = \Delta \left(\frac{1}{r} - \frac{1}{r_1} \right). \quad (5)$$

From (3) and (5),

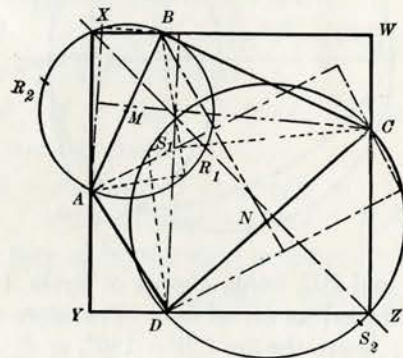
$$a = \sqrt{rr_1r_2r_3} \left(\frac{r_2 + r_3}{r_2r_3} \right) = \sqrt{rr_1r_2r_3} \left(\frac{r_1 - r}{rr_1} \right).$$

$$\begin{aligned} \text{Similarly,} \quad & \therefore a^2 = (r_2 + r_3)(r_1 - r) \\ \text{and} \quad & b^2 = (r_3 + r_1)(r_2 - r) \\ & c^2 = (r_1 + r_2)(r_3 - r) \end{aligned} \quad (6)$$

* From *American Mathematical Monthly*.

Hence, sides of required Δ are mean proportionals between sums and differences of radii of inscribed and escribed circles, as indicated in (6).

35.* Let A, B, C, D be the four points in order; join them to form a quadrilateral. On two opposite sides, AB and CD , as diameters, construct the circles, M and N , respectively. On circle M take $AR_1 =$ a quadrant; on circle N take $DS_1 =$ a quadrant. Draw R_1S_1 to cut circle M at X and circle N at Z . Let AX cut DZ at Y and BX cut CZ at W . Then $XYZW$ is a square whose sides pass through the four points — A, B, C, D .



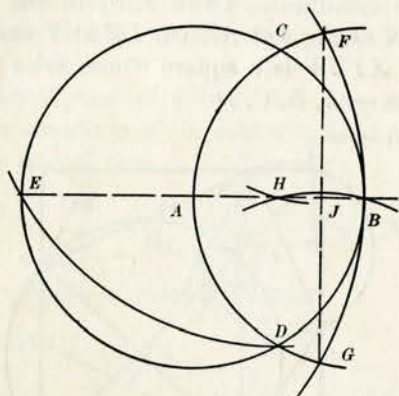
Proof. $\angle BXY$ and $\angle CZD$ are evidently rt. $\angle$ s. Further, $\angle S_1ZD = \frac{1}{2}$ rt. $\angle = \angle R_1XY$. Therefore, $\triangle XYZ$ is an isosceles right triangle. Therefore, $XYZW$ is a square.

Let R_2 denote the point diametrically opposite R_1 on circle M and, similarly, S_2 for S_1 on circle N . There are four possible diagonal lines, $R_1S_1, R_1S_2, R_2S_1, R_2S_2$, and so in general we have four distinct squares arising from the use of the circles, M and N .

* From "School Science and Mathematics."

36.* Let AB be the line segment. With A and B as centers and AB as radius describe two circles intersecting at C and D . With C as a center and CD as a radius, draw an arc intersecting circle B at E . With E as a center and EB as a radius, draw an arc intersecting circle B at F and G . With F and G as centers and FB as a radius, draw arcs intersecting at H .

Then H is the mid-point of AB .



Proof. BC and BD , being chords of circle A equal to its radius, each subtend an arc of 60° . Therefore CD subtends an arc of 120° ; hence the arc $EB = 180^\circ$, or E , A , and B are collinear and EAB is the diameter of circle A . Also FG is the common chord of the circle B and the circle whose center is E . Therefore FG is perpendicular to EB and is bisected by EB . G is equidistant from F and H by construction, and hence must lie in the line AB .

Call J the intersection point of EB and FG .

Then BJ is the altitude of $\triangle BFG$ upon FG . Also, the diameter of the circle E is $4AB$. Therefore $BF \times BG = BJ \times 4AB$. (The product of two sides of an inscribed

* From "School Science and Mathematics."

triangle equals the product of the altitude on the third side and the diameter of the circle.) Since BF and BG are each equal to AB , we have $BF \times BG = AB^2$. Hence $AB^2 = BJ \times 4AB$, or $AB = 4BJ$.

Since $\triangle FBH$ is isosceles, FJ bisects HB , or $BH = 2BJ$.

$\therefore AB = 2BH$, or H is the mid-point of AB .

38. Square (1) and

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}} = -2(y^{\frac{1}{2}}z^{\frac{1}{2}} + z^{\frac{1}{2}}x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}). \quad (3)$$

Square (3) and simplify and

$$x + y + z - 2(y^{\frac{1}{2}}z^{\frac{1}{2}} + z^{\frac{1}{2}}x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}) = 8x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}})$$

and the right member of this is 0 by (1). So

$$x + y + z = 2(y^{\frac{1}{2}}z^{\frac{1}{2}} + z^{\frac{1}{2}}x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}}). \quad (4)$$

Square (4) and simplify

$$x^2 + y^2 + z^2 - 2(yz + zx + xy) = 8x^{\frac{1}{2}}y^{\frac{1}{2}}z^{\frac{1}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}}).$$

This squared gives

$$[x^2 + y^2 + z^2 - 2(yz + zx + xy)]^2 = 64xyz[x + y + z + 2(y^{\frac{1}{2}}z^{\frac{1}{2}} + z^{\frac{1}{2}}x^{\frac{1}{2}} + x^{\frac{1}{2}}y^{\frac{1}{2}})],$$

and by help of (4) the right member of this reduces to

$$128xyz(x + y + z).$$

39. Let x and y be the required numbers.

Then

$$x - y = 3856.$$

and

$$\log x - \log y = \log \frac{x}{y} = 1.4238.$$

From log table,

$$\frac{x}{y} = 26.534.$$

Hence,

$$x = 26.534y.$$

Substituting in (1),

$$25.534y = 3856.$$

Hence,

$$y = 151.015.$$

Substituting in (1),

$$x = 4007.015.$$

40. Let x = the side of the square and ϕ be the angle between the lines x and 20.

Then $400 + x^2 - 40x \cos \phi = 900$

and $400 + x^2 + 40x \sin \phi = 1600.$

Eliminating $\sin \phi$ and $\cos \phi$, we obtain

$$x^4 - 2500x^2 = 717,500.$$

$$\therefore x = 20.073, \text{ or } 45.793.$$

It can easily be shown that the point must be inside the square for $x = 45.793$ and outside for $x = 20.073$.

41. Suppose that I am now x years old; then my mother is $4x$ years old; and in 4 years my father will be $4(x+4)$, so that he is now $4(x+4) - 4$ in age.

When I was born, my mother was $4x - x$, or $3x$, my sister was $\frac{3x}{4}$, and she is now $x + \frac{3x}{4}$ years.

My father is now $3\left(x + \frac{3x}{4}\right)$ and he is also $4(x+4) - 4$.

Hence $4x + 12 = \frac{21x}{4}$; $5x = 48$. The ages now are: mother 38.4, father 50.4, sister 16.8, I 9.6.

42. Let x = B's age when A was 6 yrs. older than B is now.

$2x$ = A's age now.

y = number yrs. until B will be 6 yrs. older than A is now.

Then $2x + 6$ = B's age in y years from now.

$2x + y$ = A's age in y years from now.

$46 + y$ = C's age in y years from now.

$$2x + 6 + 2x + y = 46 + y.$$

$$4x = 40.$$

$$2x = 20 = \text{A's age now.}$$

Now let w = number years since B was 10.

Then $10 + w + 6$ = A's age when B was 10.

$y = 16$ = number yrs. from time B was 10 until he is 26.

$$10 + w + 6 + y + 26 = 46 + y - w.$$

$$2w = 4.$$

$$\therefore w = 2$$

and $10 + w = 12$ = B's age now.

Therefore, A is now 20 yrs. old and B is 12.

43. From $x = \sqrt[x]{y}$, $x^x = y$, and from $\sqrt[x]{y} = y^{\frac{x}{y}}$,

$$\frac{1}{x} = \frac{x}{y} \text{ and } y = x^2.$$

$$\therefore x^x = x^2, \text{ whence } x = 2 \text{ and } \therefore y = 4.$$

Let d denote the distance AB . Then

$$\frac{d}{2} + \frac{d}{4} = \frac{15}{2}, \text{ whence } d = 10.$$

44. If x denotes one base and $\frac{x}{10}$ the other base, and n any positive number, then from the conditions of the problem

$$\log_{10} n = \log_x n + \log_{\frac{x}{10}} n; \text{ but } \log_x n = \frac{\log_{10} n}{\log_{10} x}.$$

Hence

$$\log_{10} n = \frac{\log_{10} n}{\log_{10} x} + \frac{\log_{10} n}{\log_{10} \frac{x}{n}}$$

$$\text{or } 1 = \frac{1}{(\log_{10} x)} + \frac{1}{(\log_{10} x - 1)}$$

or

$$(\log_{10} x)^2 - 3 \log_{10} x + 1 = 0; \log_{10} x = 2.6180 \text{ or } .3820.$$

The bases are approximately 415 and 41.5, or 2.41 and .241.

45.* Let x = time it takes Mrs. A to go from one house to the other.

Let y = time it takes Mrs. B to go from one house to the other.

If when Mrs. A arrives at the home of Mrs. B she finds that Mrs. B has been gone 11 minutes, and since it takes Mrs. A $\frac{x}{2}$ minutes to walk half of the distance back, then $11 + \frac{x}{2}$ is the number of minutes needed by Mrs. B to walk from one house to the other and half way back. Therefore

$$\frac{3y}{2} = 11 + \frac{x}{2}$$

Similarly,
$$\frac{3x}{2} = 15 + \frac{y}{2}$$

The solution is $x = 14$, $y = 12$. It takes Mrs. A 14 minutes to walk the distance and Mrs. B 12 minutes.

46. Let a = my present age, s = my sister's age, m = mother's age, and f = father's age.

Then $\frac{(m-a)}{4} = s - a$; $3s = f$;

$$\frac{(f+4)}{4} = a + 4$$
; $m = 4a$;

$$f + 4 = 4\left(\frac{m}{4} + 4\right)$$
; $f - m = 12$;

$$m - \frac{m}{4} = 4\left(\frac{f}{3} - \frac{m}{4}\right)$$
; $21m = 16f$.

$$\therefore m = 38\frac{2}{3}$$
, and $f = 50\frac{2}{3}$.

Father is 20 years older than mother, and both have the same birthday.

* From "School Science and Mathematics."

47. Assume $1 + 3 + 5 + \dots + (2k-1) = k^2$. (1)

Adding $(2k+1)$ to both members of (1) gives

$$1 + 3 + 5 + \dots + (2k-1) + (2k+1) = k^2 + (2k+1) \\ = (k+1)^2. \quad (2)$$

Hence if (1) is true for k , it is true for $(k+1)$. But (1) is true for $k=3$, therefore it is true for $k=4$. Therefore the sum of the above series is a perfect square.

48.* The figure is a top view of the embankment.

The area of the upper base, B_1 , is 61,200 sq. ft.

The area of the lower base, B , is 129,600 sq. ft.

The area of the mid-section, M , is 96,525 sq. ft.

The altitude of the solid is 11 ft. Hence the volume of the solid is $\frac{h(B_1 + B + 4M)}{6} = 1,057,650$ cu. ft.

49.* Let a = number of nuts each sailor got at last division.

Then $4a + 1$ = number fourth sailor left.

$$\frac{4}{3}(4a + 1) + 1 = \frac{16a + 7}{3} = \text{number third sailor left.}$$

$$\frac{4}{3}\left(\frac{16a + 7}{3}\right) + 1 = \frac{64a + 37}{9} = \text{number second sailor left, etc.}$$

Thus continuing we find

$$N = \frac{1024a + 781}{81} = \text{number in original pile}$$

or

$$N = 12a + 9 + \frac{52a + 52}{81},$$

the last term of which must be an integer, as all the others are.

So let

$$b = \frac{52a + 52}{81},$$

giving

$$a = b - 1 + \frac{29b}{52}.$$

* From "School Science and Mathematics."

As three terms of this are integers, b must be a multiple of 52.

So let

$$b = 52 X.$$

Substituting in the previous equation,

$$a = 81 X - 1,$$

and putting this into the equation for N :

$$N = 1024 X - 3,$$

where X may be taken arbitrarily.

50.* Let x = number of coconuts.

If there had been $(x + 4)$, the first sailor could have taken one more and left four more than he did. Then the second could have taken one more and left four more than he did. And so on to the fifth. In this case there would have been none for the monkey, but a live monkey on a coconut island is not in need of pity. The number that would remain for distribution in the morning is greater by 4 than an exact multiple of 5, is then $(10y + 4)$ or $(10y + 9)$ where y is some positive integer. But this number is equal to $(x + 4)$ multiplied by (four-fifths)⁵. Consequently

$$1024 \frac{(x + 4)}{3125} = 10y + 4, \text{ or } 10y + 9.$$

The $(10y + 9)$ is barred because it would lead to the absurdity: even = odd.

Use of the other leads to the equation

$$512x - 15,625y = 4202$$

which must be solved in positive integers. Standard algebraic methods lead to the solutions

$$x = 3121 + 15,625t,$$

$$y = 102 + 512t,$$

where $t = 0, 1, 2, 3, \dots$

* From "School Science and Mathematics."

51. Let $5x$ = radius of the top.

$3x$ = radius of the bottom.

Since the kettle is a frustum of a cone and contains 139 gallons, or 3003 cubic inches, we have

$$4(25\pi x^2 + 9\pi x^2 + 15\pi x^2) = 3003.$$

$$\therefore 196\pi x^2 = 3003.$$

Solving,

$$x = 2.208.$$

$\therefore 5x = 11.04$ inches and $3x = 6.624$ inches.

52. Volume of ball = volume of cylindrical rod.

Volume of ball = $\frac{4}{3}\pi \cdot 24^3$, or 2304π cu. in.

Surface of ball = $\pi \cdot 24^2$, or 576π sq. in.

Area of end of rod = $\pi(\frac{9}{2})^2$, or 9π .

Length of rod = $2304\pi \div 9\pi$, or 256 in.

Convex surface of rod = $6\pi \cdot 256$, or 1536π sq. in.

Basal surface = $9\pi \cdot 2$, or 18π sq. in.

Surface of rod = $1536\pi + 18\pi$, or 1554π sq. in.

By proportion we have,

$$576\pi : 1554\pi = 1 \text{ oz.} : x \text{ oz.}$$

$\therefore x = 2.698$ oz., the number of ounces of gold required to plate the rod.

NUTS AND KERNELS

1. If an article had cost me 10% less, my rate of gain would have been 3 times as great. What is my per cent gain?

<i>Solution.</i>	Let 100 = cost
and	x = the number of per cent gained.
Then	$100 + x$ = the selling price.
Also	90 = the supposed cost
and	$10 + x$ = the supposed gain.

Now $\frac{x}{100} =$ the per cent gained
 and $\frac{10+x}{90} =$ the supposed per cent of gain.
 $\therefore 3\left(\frac{x}{100}\right) = \frac{10+x}{90}$.
 $\therefore x = 5\frac{1}{3}$, the number of per cent gained.

2. If 2 were 3, what would 3 be?

Solution. The true value of the 1st is to its assumed value as the true value of the 2nd is to its assumed value.

$\therefore 2:3 = 3:(4\frac{1}{2})$. $\therefore 3$ would be $4\frac{1}{2}$.

3. What two equal payments, one now and the other in two years, will settle a bill of \$250, with interest at 6%?

Solution. The amount of \$250 for 2 yrs. at 6% = \$280. For each dollar in the equal payments there must be paid \$1.00 + \$1.12, or \$2.12. Hence the required payment is $\$280 \div 2.12$, or \$132.07+.

4. B can saw $2\frac{1}{2}$ cords of wood per day or split $3\frac{1}{2}$ cords already sawed. How much must he saw so that he may be occupied the rest of the day in splitting it?

Solution. B can saw $5\frac{1}{2}$ cords in one day. To saw one cord it takes him $\frac{2}{5}$ of a day. B can split $\frac{7}{2}$ cords in one day. To split a cord it takes him $\frac{2}{7}$ of a day. To saw and split one cord it takes him $\frac{2}{5} + \frac{2}{7}$, or $\frac{24}{35}$ da. In $\frac{1}{35}$ of a day he can saw and split $\frac{1}{24}$ of a cord. In $\frac{35}{24}$, or one day, he can saw and split $35 \times \frac{1}{24}$ of a cord, or $1\frac{1}{24}$ cords.

5. I bought eggs at 10 cents a dozen. Had I bought 3 more for the same sum they would have cost me 3 cents a dozen less. How many eggs did I buy?

Solution.

Let $x =$ the number.

Then

$$\frac{10}{12}x = \frac{7}{12}(x+3)$$

$\therefore x = 7$, the number of eggs.

QUOTATIONS *

"There is no prophet which preaches the superpersonal God more plainly than Mathematics." — PAUL CARUS.

"And Reason now through number, time, and space
 Darts the keen lustre of her serious eye;
 And learns from facts compar'd the laws to trace
 Whose long procession leads to Deity."

— JAMES BEATTIE.

"Mathematics, too, is a language, and as concerns its structure and content it is the most perfect language which exists, superior to any vernacular; indeed, since it is understood by every people, mathematics may be called the language of languages. Through it, as it were, nature herself speaks; through it the Creator of the world has spoken, and through it the Preserver of the world shall continue to speak." — C. DILLMAN.

"Why are wise few, fools numerous in the excesse?

'Cause, wanting number, they are numberlesse."

— LOVELACE.

"Once when lecturing to a class he (Lord Kelvin) used the word 'Mathematician,' and then interrupting himself asked his class: 'Do you know what a mathematician is?' Stepping to the blackboard he wrote upon it:

$$\int_{-\alpha}^{+\alpha} e^{-x^2} dx = \sqrt{\pi}$$

"Then putting his finger on what he had written, he turned to his class and said: 'A mathematician is one to whom

* For other Quotations see "Mathematical Wrinkles."

that is as obvious as that twice two makes four is to you.' Liouville was a mathematician." — S. P. THOMPSON.

"It is only through Mathematics that we can thoroughly understand what true science is. Here alone we can find in the highest degree simplicity and severity of scientific law, and such abstraction as the human mind can attain. Any scientific education setting forth from any other point is faulty in its basis." — A. COMTE.

"(Arithmetic) is another of the great master-keys of life. With it the astronomer opens the depths of the heavens; the engineer, the gates of the mountains; the navigator, the pathways of the deep. The skillful arrangement, the rapid handling of figures, is a perfect magician's wand. The mighty commerce of the United States, foreign and domestic, passes through the books kept by some thousands of diligent and faithful clerks. Eight hundred bookkeepers, in the Bank of England, strike the monetary balance of half the civilized world. Their skill and accuracy in applying the common rules of arithmetic are as important as the enterprise and capital of the merchant, or the industry and courage of the navigator. I look upon a well-kept ledger with something of the pleasure with which I gaze on a picture or a statue. It is a beautiful work of art." — EDWARD EVERETT.

"Someone who had begun to read geometry with Euclid, when he had learned the first proposition, asked Euclid, 'But what shall I get by learning these things?' Whereupon Euclid called his slave and said, 'Give him three-pence, since he must make gain out of what he learns.'" — STOBÆUS.

"Leibnitz believed he saw the image of creation in his binary arithmetic in which he employed only two characters, unity and zero. Since God may be represented by unity,

and nothing by zero, he imagined that the Supreme Being might have drawn all things from nothing, just as in the binary arithmetic all numbers are expressed by unity with zero. This idea was so pleasing to Leibnitz, that he communicated it to the Jesuit Grimaldi, President of the Mathematical Board of China, with the hope that this emblem of the creation might convert to Christianity the reigning emperor who was particularly attached to the sciences."

— LAPLACE.

"Following the example of Archimedes, who wished his tomb decorated with his most beautiful discovery in geometry and ordered it inscribed with a cylinder circumscribed by a sphere, James Bernoulli requested that his be inscribed with his logarithmic spiral together with the words, 'Eadem mutata resurgo,' a happy allusion to the hope of the Christians, which is in a way symbolized by the properties of that curve." — FONTENELLE.

"Lord Kelvin, unable to meet his classes one day, posted the following notice on the door of his lecture room, —

'Professor Thomson will not meet his classes today.' The disappointed class decided to play a joke on the professor. Erasing the 'c,' they left the legend to read, —

'Professor Thomson will not meet his asses today.' When the class assembled the next day in anticipation of the effect of their joke, they were astonished and chagrined to find that the professor had outwitted them. The legend of yesterday was now found to read, —

'Professor Thomson will not meet his asses today.'" —

— CYRUS NORTHUP.

"I was at the mathematical school, where the master taught his pupils, after a method, scarce imaginable to us in Europe. The propositions, and demonstrations, were

fairly written on a thin wafer, with ink composed of a cephalic tincture. This, the student was to swallow upon a fasting stomach, and for three days following, eat nothing but bread and water. As the wafer digested, the tincture mounted to his brain, bearing the proposition along with it. But the success has not hitherto been answerable, partly by the perverseness of lads; to whom this bolus is so nauseous, that they generally steal aside, and discharge it upwards, before it can operate; neither have they been yet persuaded to use so long an abstinence as the prescription requires."

— JONATHAN SWIFT.

"A pseudomath is a person who handles mathematics as a monkey handles the razor. The creature tried to shave himself as he had seen his master do; but, not having any notion of the angle at which the razor was to be held, he cut his own throat. He never tried it a second time, poor animal! but the pseudomath keeps on in his work, proclaims himself clean shaved, and all the rest of the world hairy.

"The graphomath is a person who, having no mathematics, attempts to describe a mathematician. Novelists perform in this way: even Walter Scott now and then burns his fingers. His dreaming calculator, Davy Ramsay, swears 'by the bones of the immortal Napier.' Scott thought that the philomaths worshipped relics: so they do in one sense."

— A. DE MORGAN.

"Alexander is said to have asked Menaechmus to teach him Geometry concisely, but Menaechmus replied: 'O king, through the country there are royal roads and roads for common citizens, but in geometry there is one road for all.'"

— STOBÆUS.

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"In Mathematicks he was greater
 Than Tycho Brahe, or Erra Pater :
 For he, by Geometrick scale,
 Could take the size of Pots of Ale ;
 Resolve by sines and tangents, straight,
 If Bread or Butter wanted weight ;
 And wisely tell what hour o' th' day
 The clock doth strike, by Algebra."

— SAMUEL BUTLER.

The End.